3 Stress and the Balance Principles

Three basic laws of physics are discussed in this Chapter:

(1) The Law of Conservation of Mass
(2) The Balance of Linear Momentum
(3) The Balance of Angular Momentum

together with the conservation of mechanical energy and the principle of virtual work, which are different versions of (2).

(2) and (3) involve the concept of stress, which allows one to describe the action of forces in materials.

3.1 Conservation of Mass

3.1.1 Mass and Density

Mass is a non-negative scalar measure of a body's tendency to resist a change in motion.

Consider a small volume element Δv whose mass is Δm. Define the average **density** of this volume element by the ratio

$$\rho_{\text{AVE}} = \frac{\Delta m}{\Delta v} \tag{3.1.1}$$

If p is some point within the volume element, then define the **spatial mass density** at p to be the limiting value of this ratio as the volume shrinks down to the point,

$$\boxed{\rho(\mathbf{x},t) = \lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v}} \qquad \textbf{Spatial Density} \tag{3.1.2}$$

In a real material, the incremental volume element Δv must not actually get too small since then the limit ρ would depend on the atomistic structure of the material; the volume is only allowed to decrease to some minimum value which contains a large number of molecules. The spatial mass density is a representative average obtained by having Δv large compared to the atomic scale, but small compared to a typical length scale of the problem under consideration.

The density, as with displacement, velocity, and other quantities, is defined for *specific particles* of a continuum, and is a continuous function of coordinates and time, $\rho = \rho(\mathbf{x},t)$. However, the mass is not defined this way – one writes for the mass of an infinitesimal volume of material – a **mass element**,

$$dm = \rho(\mathbf{x},t)dv \tag{3.1.3}$$

or, for the mass of a volume v of material at time t,

$$m = \int_v \rho(\mathbf{x},t)dv \tag{3.1.4}$$

3.1.2 Conservation of Mass

The law of conservation of mass states that mass can neither be created nor destroyed.

Consider a collection of matter located somewhere in space. This quantity of matter with well-defined boundaries is termed a **system**. The law of conservation of mass then implies that the mass of this given system remains constant,

$$\boxed{\frac{Dm}{Dt} = 0} \qquad \textbf{Conservation of Mass} \qquad (3.1.5)$$

The volume occupied by the matter may be changing and the density of the matter within the system may be changing, but the mass remains constant.

Considering a differential mass element at position **X** in the reference configuration and at **x** in the current configuration, Eqn. 3.1.5 can be rewritten as

$$dm(\mathbf{X}) = dm(\mathbf{x}, t) \qquad (3.1.6)$$

The conservation of mass equation can be expressed in terms of densities. First, introduce ρ_0, the **reference mass density** (or simply the **density**), defined through

$$\boxed{\rho_0(\mathbf{X}) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}} \qquad \textbf{Density} \qquad (3.1.7)$$

Note that the density ρ_0 and the spatial mass density ρ are *not* the same quantities[1].

Thus the **local** (or **differential**) **form** of the conservation of mass can be expressed as (see Fig. 3.1.1)

$$dm = \rho_0(\mathbf{X})dV = \rho(\mathbf{x}, t)dv = \text{const} \qquad (3.1.8)$$

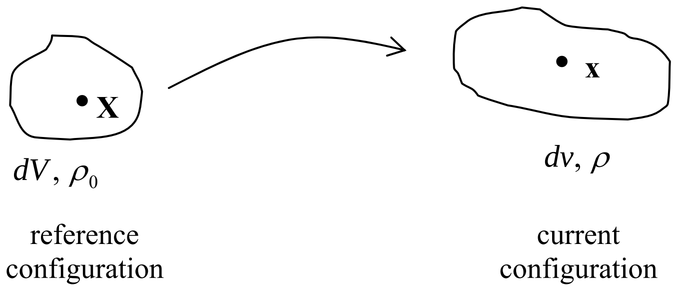


Figure 3.1.1: Conservation of Mass for a deforming mass element

Integration over a finite region of material gives the **global** (or **integral**) **form**,

$$m = \int_V \rho_0(\mathbf{X})dV = \int_v \rho(\mathbf{x}, t)dv = \text{const} \qquad (3.1.9)$$

or

$$\dot{m} = \frac{dm}{dt} = \frac{d}{dt}\int_v \rho(\mathbf{x}, t)dv = 0 \qquad (3.1.10)$$

[1] they not only are functions of different variables, but also have different values; they are not different representations of the same thing, as were, for example, the velocities **v** and **V**. One could introduce a material mass density, $\mathrm{P}(X, t) = \rho(x(X, t), t)$, but such a quantity is not useful in analysis

3.1.3 Control Mass and Control Volume

A **control mass** is a *fixed mass* of material whose volume and density may change, and which may move through space, Fig. 3.1.2. There is no mass transport through the moving surface of the control mass. For such a system, Eqn. 3.1.10 holds.

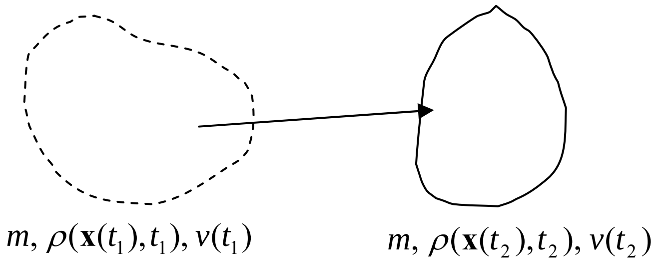


Figure 3.1.2: Control Mass

By definition, the derivative in 3.1.10 is the time derivative of a property (in this case mass) of a collection of material particles as they move through space, and when they instantaneously occupy the volume *v*, Fig. 3.1.3, or

$$\frac{d}{dt}\int_{dv}\rho\, dv = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{v(t+\Delta t)}\rho(\mathbf{x},t+\Delta t)dv - \int_{v(t)}\rho(\mathbf{x},t)dv\right\} = 0 \qquad (3.1.11)$$

Alternatively, one can take the material derivative inside the integral sign:

$$\frac{dm}{dt} = \int_v \frac{d}{dt}\left[\rho(\mathbf{x},t)dv\right] = 0 \qquad (3.1.12)$$

This is now equivalent to the sum of the rates of change of mass of the mass elements occupying the volume *v*.

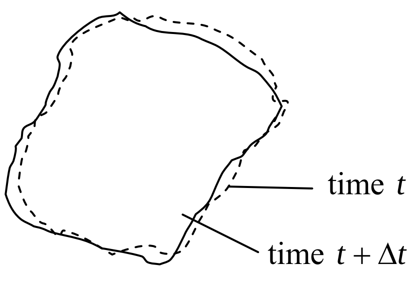


Figure 3.1.3: Control Mass occupying different volumes at different times

A **control volume**, on the other hand, is a *fixed volume* (region) of space through which material may flow, Fig. 3.1.4, and for which *the mass may change*. For such a system, one has

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\int_v \rho(\mathbf{x},t)dv = \int_v \frac{\partial}{\partial t}\left[\rho(\mathbf{x},t)\right]dv \neq 0 \tag{3.1.13}$$

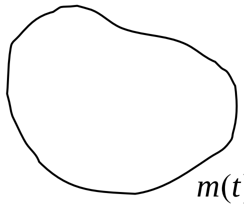


Figure 3.1.4: Control Volume

3.1.4 The Continuity Equation (Spatial Form)

A consequence of the law of conservation of mass is the **continuity equation**, which (in the spatial form) relates the density and velocity of any material particle during motion. This equation can be derived in a number of ways:

Derivation of the Continuity Equation using a Control Volume (Global Form)

The continuity equation can be derived directly by considering a control volume - this is the derivation appropriate to fluid mechanics. Mass inside this fixed volume cannot be created or destroyed, so that the rate of increase of mass in the volume must equal the rate at which mass is flowing into the volume through its bounding surface.

The rate of increase of mass inside the fixed volume v is

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\int_v \rho(\mathbf{x},t)\,dv = \int_v \frac{\partial \rho}{\partial t}dv \tag{3.1.14}$$

The **mass flux** (rate of flow of mass) out through the surface is given by Eqn. 1.7.9,

$$\int_s \rho\mathbf{v}\cdot\mathbf{n}ds, \qquad \int_s \rho v_i n_i\, ds$$

where **n** is the unit outward normal to the surface and **v** is the velocity. It follows that

$$\int_v \frac{\partial \rho}{\partial t}dv + \int_s \rho\mathbf{v}\cdot\mathbf{n}ds = 0, \qquad \int_v \frac{\partial \rho}{\partial t}dv + \int_s \rho v_i n_i\, ds = 0 \tag{3.1.15}$$

Use of the divergence theorem 1.7.12 leads to

$$\int_v \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{v})\right]dv = 0, \qquad \int_v \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i}\right]dv = 0 \tag{3.1.16}$$

leading to the continuity equation,

$$\begin{array}{ll} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 & \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v_i)}{\partial x_i} = 0 \\ \dfrac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0 & \dfrac{d\rho}{dt} + \rho \dfrac{\partial v_i}{\partial x_i} = 0 \\ \dfrac{\partial \rho}{\partial t} + \operatorname{grad}\rho \cdot \mathbf{v} + \rho \operatorname{div}\mathbf{v} = 0 & \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial x_i} v_i + \rho \dfrac{\partial v_i}{\partial x_i} = 0 \end{array}$$

Continuity Equation

(3.1.17)

This is (these are) the continuity equation in spatial form. The second and third forms of the equation are obtained by re-writing the local derivative in terms of the material derivative 2.4.7 (see also 1.6.23b).

If the material is incompressible, so the density remains constant in the neighbourhood of a particle as it moves, then the continuity equation reduces to

$$\operatorname{div}\mathbf{v} = 0, \quad \frac{\partial v_i}{\partial x_i} = 0$$

Continuity Eqn. for Incompressible Material (3.1.18)

Derivation of the Continuity Equation using a Control Mass

Here follow two ways to derive the continuity equation using a control mass.

1. Derivation using the Formal Definition

From 3.1.11, adding and subtracting a term:

$$\begin{aligned} \frac{d}{dt}\int_{dv} \rho \, dv = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Bigg\{ & \left[\int_{v(t+\Delta t)} \rho(\mathbf{x}, t+\Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t+\Delta t) dv \right] \\ & + \left[\int_{v(t)} \rho(\mathbf{x}, t+\Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t) dv \right] \Bigg\} \end{aligned} \tag{3.1.19}$$

The terms in the second square bracket correspond to holding the volume v fixed and evidently equals the local rate of change:

$$\frac{d}{dt}\int_{dv} \rho \, dv = \int_{v} \frac{\partial \rho}{\partial t} dv + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{v(t+\Delta t)-v(t)} \rho(\mathbf{x}, t+\Delta t) dv \tag{3.1.20}$$

The region $v(t+\Delta t) - v(t)$ is swept out in time Δt. Superimposing the volumes $v(t)$ and $v(t+\Delta t)$, Fig. 3.1.5, it can be seen that a small element Δv of $v(t+\Delta t) - v(t)$ is given by (see the example associated with Fig. 1.7.7)

$$\Delta v = \Delta t \mathbf{v} \cdot \mathbf{n} \Delta s \tag{3.1.21}$$

where s is the surface. Thus

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{v(t+\Delta t)-v(t)} \rho(\mathbf{x}, t+\Delta t) dv = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_s \Delta t \rho(\mathbf{x}, t+\Delta t) \mathbf{v} \cdot \mathbf{n} ds = \int_s \rho(\mathbf{x}, t) \mathbf{v} \cdot \mathbf{n} ds \quad (3.1.22)$$

and 3.1.15 is again obtained, from which the continuity equation results from use of the divergence theorem.

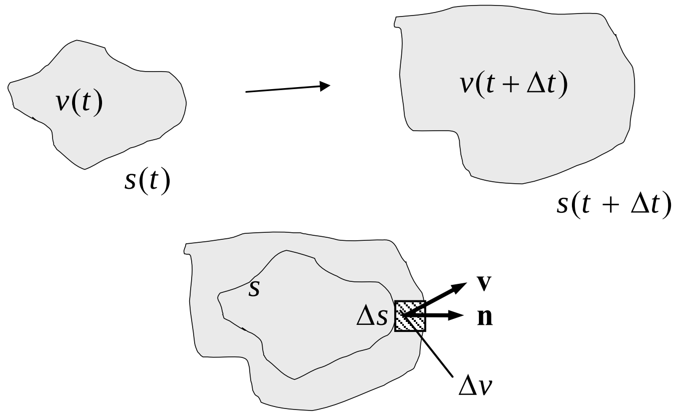


Figure 3.1.5: Evaluation of Eqn. 3.1.22

2. Derivation by Converting to Mass Elements

This derivation requires the kinematic relation for the material time derivative of a volume element, 2.5.23: $d(dv)/dt = \operatorname{div}\mathbf{v}\, dv$. One has

$$\frac{dm}{dt} = \frac{d}{dt} \int_v \rho(\mathbf{x}, t) dv = \int_v \frac{d}{dt} (\rho dv) = \int_v \left(\dot{\rho} dv + \rho \overline{\dot{dv}} \right) = \int_v (\dot{\rho} + \operatorname{div}\mathbf{v} \rho) dv \equiv 0 \quad (3.1.23)$$

The continuity equation then follows, since this must hold for any arbitrary region of the volume v.

Derivation of the Continuity Equation using a Control Volume (Local Form)

The continuity equation can also be derived using a differential control volume element. This calculation is similar to that given in §1.6.6, with the velocity $\mathbf{v}$ replaced by $\rho\mathbf{v}$.

3.1.5 The Continuity Equation (Material Form)

From 3.1.9, and using 2.2.53, $dv = J dV$,

$$\int_V [\rho_0(\mathbf{X}) - \rho(\boldsymbol{\chi}(\mathbf{X}, t), t) J(\mathbf{X}, t)] dV = 0 \quad (3.1.24)$$

Since V is an arbitrary region, the integrand must vanish everywhere, so that

$$\boxed{\rho_0(\mathbf{X}) = \rho(\chi(\mathbf{X},t),t)J(\mathbf{X},t)} \quad \textbf{Continuity Equation (Material Form)} \qquad (3.1.25)$$

This is known as the continuity (mass) equation in the material description. Since $\dot{\rho}_0 = 0$, the rate form of this equation is simply

$$\frac{d}{dt}(\rho J) = 0 \qquad (3.1.26)$$

The material form of the continuity equation, $\rho_0 = \rho J$, is an algebraic equation, unlike the partial differential equation in the spatial form. However, the two must be equivalent, and indeed the spatial form can be derived directly from this material form: using 2.5.20, $dJ/dt = J\operatorname{div}\mathbf{v}$,

$$\begin{aligned} \frac{d}{dt}(\rho J) &= \dot{\rho}J + \rho\dot{J} \\ &= J\left(\dot{\rho} + \rho\operatorname{div}\mathbf{v}\right) \end{aligned} \qquad (3.1.27)$$

This is zero, and $J > 0$, and the spatial continuity equation follows.

Example (of Conservation of Mass)

Consider a bar of material of length l_0, with density in the undeformed configuration ρ_0 and spatial mass density $\rho(x,t)$, undergoing the 1-D motion $\mathbf{X} = \mathbf{x}/(1+At)$, $\mathbf{x} = \mathbf{X} + At\mathbf{X}$. The volume ratio (taking unit cross-sectional area) is $J = 1 + At$. The continuity equation in the material form 3.1.25 specifies that

$$\rho_0 = \rho(1 + At)$$

Suppose now that

$$\rho_o(\mathbf{X}) = \frac{2m}{l_0^2}\mathbf{X}$$

so that the total mass of the bar is $\int_0^{l_o} \rho_0(\mathbf{X})d\mathbf{X} = m$. It follows that the spatial mass density is

$$\rho = \frac{\rho_0}{(1+At)} = \frac{2m}{l_0^2}\frac{\mathbf{X}}{1+At} = \frac{2m}{l_0^2}\frac{\mathbf{x}}{(1+At)^2}$$

Evaluating the total mass of the bar at time *t* leads to

$$\int_0^{l_o(1+At)} \rho(\mathbf{x},t)\,d\mathbf{x} = \frac{2m}{l_0^2}\frac{1}{(1+At)^2}\int_0^{l_o(1+At)} \mathbf{x}\,d\mathbf{x}$$

which is again *m*, as required.

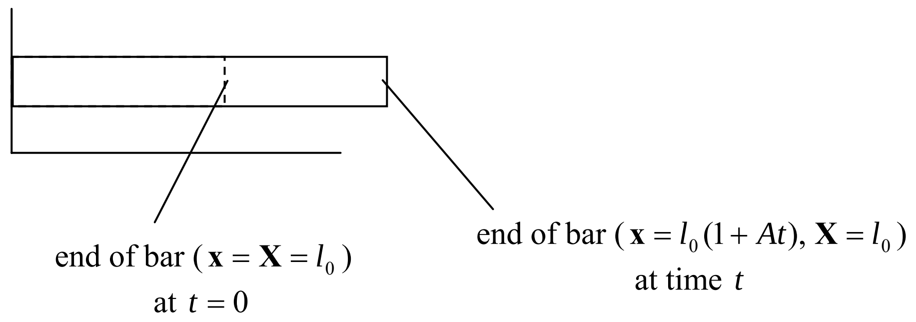


Figure 3.1.6: a stretching bar

The density could have been derived from the equation of continuity in the spatial form: since the velocity is

$$\mathbf{V}(\mathbf{X},t) = \frac{d\mathbf{x}(\mathbf{X},t)}{dt} = A\mathbf{X}, \qquad \mathbf{v}(\mathbf{x},t) = \mathbf{V}(\chi^{-1}(\mathbf{x},t),t) = \frac{A\mathbf{x}}{1+At}$$

one has

$$\frac{\partial\rho}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial\mathbf{x}} + \rho\frac{\partial v}{\partial\mathbf{x}} = \frac{\partial\rho}{\partial t} + \frac{A\mathbf{x}}{1+At}\frac{\partial\rho}{\partial\mathbf{x}} + \rho\frac{A}{1+At} = 0$$

Without attempting to solve this first order partial differential equation, it can be seen by substitution that the value for ρ obtained previously satisfies the equation.

■

3.1.6 Material Derivatives of Integrals

Reynold's Transport Theorem

In the above, the material derivative of the total mass carried by a control mass,

$$\frac{d}{dt}\int_v \rho(\mathbf{x},t)dv,$$

was considered. It is quite often that one needs to evaluate material time derivatives of similar volume (and line and surface) integrals, involving other properties, for example momentum or energy. Thus, suppose that $\mathbf{A}(\mathbf{x},t)$ is the distribution of some property (per unit volume) throughout a volume *v* (**A** is taken to be a second order tensor, but what follows applies also to vectors and scalars). Then the rate of change of the total amount of the property carried by the mass system is

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv$$

Again, this integral can be evaluated in a number of ways. For example, one could evaluate it using the formal definition of the material derivative, as done above for $\mathbf{A} = \rho$. Alternatively, one can evaluate it using the relation 2.5.23, $d(dv)/dt = \text{div}\mathbf{v}\, dv$, through

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv = \int_v \frac{d}{dt}\left[\mathbf{A}(\mathbf{x},t)dv\right] = \int_v \left[\dot{\mathbf{A}}dv + \mathbf{A}\,\dot{\overline{dv}}\right] = \int_v \left[\dot{\mathbf{A}} + \text{div}\mathbf{v}\mathbf{A}\right]dv \qquad (3.1.28)$$

Thus one arrives at **Reynold's transport theorem**

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv = \begin{cases} \displaystyle\int_v \left[\frac{d\mathbf{A}}{dt} + \text{div}\mathbf{v}\mathbf{A}\right]dv & \displaystyle\int_v \left[\frac{dA_{ij}}{dt} + \frac{\partial v_k}{\partial x_k}A_{ij}\right]dv \\ \displaystyle\int_v \left[\frac{\partial \mathbf{A}}{\partial t} + \text{grad}\mathbf{A}\cdot\mathbf{v} + \text{div}\mathbf{v}\mathbf{A}\right]dv & \displaystyle\int_v \left[\frac{\partial A_{ij}}{\partial t} + \frac{\partial A_{ij}}{\partial x_k}v_k + \frac{\partial v_k}{\partial x_k}A_{ij}\right]dv \\ \displaystyle\int_v \left[\frac{\partial \mathbf{A}}{\partial t} + \text{div}(\mathbf{A}\otimes\mathbf{v})\right]dv & \displaystyle\int_v \left[\frac{\partial A_{ij}}{\partial t} + \frac{\partial (A_{ij}v_k)}{\partial x_k}\right]dv \\ \displaystyle\int_v \frac{\partial \mathbf{A}}{\partial t}dv + \int_s \mathbf{A}(\mathbf{v}\cdot\mathbf{n})ds & \displaystyle\int_v \frac{\partial A_{ij}}{\partial t}dv + \int_s A_{ij}v_k n_k ds \end{cases}$$

Reynold's Transport Theorem (3.1.29)

The index notation is shown for the case when **A** is a second order tensor. In the last of these forms[2] (obtained by application of the divergence theorem), the first term represents the amount (of **A**) created within the volume *v* whereas the second term (the flux term) represents the (volume) rate of flow of the property through the surface. In the last three versions, Reynold's transport theorem gives the material derivative of the moving control mass in terms of the derivative of the instantaneous fixed volume in space (the first term).

Of course when $\mathbf{A} = \rho$, the continuity equation is recovered.

Another way to derive this result is to first convert to the reference configuration, so that integration and differentiation commute (since dV is independent of time):

$$\begin{aligned}\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv &= \frac{d}{dt}\int_V \mathbf{A}(\mathbf{X},t)JdV = \int_V \frac{d}{dt}\left(\mathbf{A}(\mathbf{X},t)J\right)dV \\ &= \int_V \left(\dot{\mathbf{A}}J + \mathbf{A}\dot{J}\right)dV = \int_V \left(\dot{\mathbf{A}} + \text{div}\mathbf{v}\mathbf{A}\right)JdV \\ &= \int_v \left(\dot{\mathbf{A}}(\mathbf{x},t) + \text{div}\mathbf{v}\mathbf{A}(\mathbf{x},t)\right)dv\end{aligned} \qquad (3.1.30)$$

[2] also known as the **Leibniz formula**

Reynold's Transport Theorem for Specific Properties

A property that is given per unit mass is called a **specific property**. For example, specific heat is the heat per unit mass. Consider then a property **B**, a scalar, vector or tensor, which is defined per unit mass through a volume. Then the rate of change of the total amount of the property carried by the mass system is simply

$$\frac{d}{dt}\int_v \rho\mathbf{B}(\mathbf{x},t)dv = \int_v \frac{d}{dt}\left[\mathbf{B}\rho dv\right] = \int_v \frac{d}{dt}\left[\mathbf{B}dm\right] = \int_v \frac{d\mathbf{B}}{dt}dm = \int_v \rho\frac{d\mathbf{B}}{dt}dv \qquad (3.1.31)$$

Material Derivatives of Line and Surface Integrals

Material derivatives of line and surface integrals can also be evaluated. From 2.5.8, $d(d\mathbf{x})/dt = \mathbf{l}d\mathbf{x}$,

$$\frac{d}{dt}\int \mathbf{A}(\mathbf{x},t)d\mathbf{x} = \int\left[\dot{\mathbf{A}} + \mathbf{A}\mathbf{l}\right]d\mathbf{x} \qquad (3.1.32)$$

and, using 2.5.22, $d(\hat{\mathbf{n}}ds)/dt = \left(\text{div}\mathbf{v} - \mathbf{l}^{\text{T}}\right)\hat{\mathbf{n}}ds$,

$$\frac{d}{dt}\int_s \mathbf{A}(\mathbf{x},t)\hat{\mathbf{n}}ds = \int_s\left[\dot{\mathbf{A}} + \mathbf{A}\left(\text{div}\mathbf{v} - \mathbf{l}^{\text{T}}\right)\right]\hat{\mathbf{n}}ds \qquad (3.1.33)$$

3.1.7 Problems

1. A motion is given by the equations

$$x_1 = X_1 + 3X_2 t, \quad x_2 = -X_1 t^2 + X_2(t+1), \quad x_3 = X_3$$

(a) Calculate the spatial mass density ρ in terms of the density ρ_0

(b) Derive a first order ordinary differential equation for the density ρ (in terms of **x** and t only) assuming that it is independent of position **x**

3.2 The Momentum Principles

In Parts I and II, the basic dynamics principles used were Newton's Laws, and these are equivalent to force equilibrium and moment equilibrium. For example, they were used to derive the stress transformation equations in Part I, §3.4 and the Equations of Motion in Part II, §1.1. Newton's laws there were applied to differential material elements.

An alternative but completely equivalent set of dynamics laws are **Euler's Laws**; these are more appropriate for finite-sized collections of moving particles, and can be used to express the force and moment equilibrium in terms of integrals. Euler's Laws are also called the **Momentum Principles**: the **principle of linear momentum** (Euler's first law) and the **principle of angular momentum** (Euler's second law).

3.2.1 The Principle of Linear Momentum

Momentum is a measure of the tendency of an object to keep moving once it is set in motion. Consider first the particle of rigid body dynamics: the (linear) momentum $\mathbf{p}$ is defined to be its mass times velocity, $\mathbf{p} = m\mathbf{v}$. The rate of change of momentum $\dot{\mathbf{p}}$ is

$$\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{3.2.1}$$

and use has been made of the fact that $dm/dt = 0$. Thus Newton's second law, $\mathbf{F} = m\mathbf{a}$, can be rewritten as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{3.2.2}$$

This equation, formulated by Euler, states that *the rate of change of momentum is equal to the applied force*. It is called the **principle of linear momentum**, or **balance of linear momentum**. If there are no forces applied to a system, the total momentum of the system remains constant; the law in this case is known as the **law of conservation of (linear) momentum**.

Eqn. 3.2.2 as applied to a particle can be generalized to the mechanics of a continuum in one of two ways. One could consider a differential element of material, of mass dm and velocity $\mathbf{v}$. Alternatively, one can consider a finite portion of material, a control mass in the current configuration with spatial mass density $\rho(\mathbf{x},t)$ and spatial velocity field $\mathbf{v}(\mathbf{x},t)$. The total linear momentum of this mass of material is

$$\boxed{\mathbf{L}(t) = \int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv} \quad \textbf{Linear Momentum} \tag{3.2.3}$$

The principle of linear momentum states that

$$\dot{\mathbf{L}}(t) = \frac{d}{dt}\int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv = \mathbf{F}(t) \tag{3.2.4}$$

where $\mathbf{F}(t)$ is the resultant of the forces acting on the portion of material.

Note that the volume over which the integration in Eqn. 3.2.4 takes place is not fixed; the integral is taken over a *fixed portion of material particles*, and the space occupied by this matter may change over time.

By virtue of the Transport theorem relation 3.1.31, this can be written as

$$\dot{\mathbf{L}}(t) = \int_v \rho(\mathbf{x},t)\dot{\mathbf{v}}(\mathbf{x},t)dv = \mathbf{F}(t) \qquad (3.2.5)$$

The resultant force acting on a body is due to the surface tractions **t** acting over surface elements and body forces **b** acting on volume elements, Fig. 3.2.1:

$$\boxed{\mathbf{F}(t) = \int_s \mathbf{t}ds + \int_v \mathbf{b}dv, \qquad F_i = \int_s t_i\, ds + \int_v b_i\, dv} \qquad \textbf{Resultant Force} \qquad (3.2.6)$$

and so the principle of linear momentum can be expressed as

$$\boxed{\int_s \mathbf{t}ds + \int_v \mathbf{b}dv = \int_v \rho\dot{\mathbf{v}}dv} \qquad \textbf{Principle of Linear Momentum} \qquad (3.2.7)$$

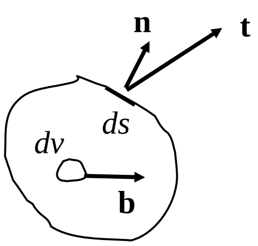


Figure 3.2.1: surface and body forces acting on a finite volume of material

The principle of linear momentum, Eqns. 3.2.7, will be used to prove Cauchy's Lemma and Cauchy's Law in the next section and, in §3.6, to derive the Equations of Motion.

3.2.2 The Principle of Angular Momentum

Considering again the mechanics of a single particle: the **angular momentum** is the moment of momentum about an axis, in other words, it is the product of the linear momentum of the particle and the perpendicular distance from the axis of its line of action. In the notation of Fig. 3.2.2, the angular momentum **h** is

$$\mathbf{h} = \mathbf{r} \times m\mathbf{v} \qquad (3.2.8)$$

which is the vector with magnitude $d \times m|\mathbf{v}|$ and perpendicular to the plane shown.

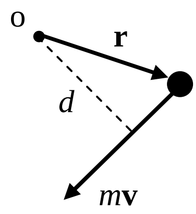


Figure 3.2.2: surface and body forces acting on a finite volume of material

Consider now a collection of particles. The **principle of angular momentum** states that the resultant moment of the external forces acting on the system of particles, $\mathbf{M}$, equals the rate of change of the total angular momentum of the particles:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt} \tag{3.2.9}$$

Generalising to a continuum, the angular momentum is

$$\boxed{\mathbf{H} = \int_v \mathbf{r} \times \rho \mathbf{v} dv} \quad \textbf{Angular Momentum} \tag{3.2.10}$$

and the principle of angular momentum is

$$\int_s \mathbf{r} \times \mathbf{t}^{(\mathbf{n})} ds + \int_v \mathbf{r} \times \mathbf{b} dv = \frac{d}{dt} \int_v \mathbf{r} \times \rho \mathbf{v} dv$$

Principle of Angular Momentum

$$\int_s \varepsilon_{ijk} x_j t_k^{(\mathbf{n})} ds + \int_v \varepsilon_{ijk} x_j b_k \, dv = \frac{d}{dt} \int_V \varepsilon_{ijk} x_j \rho v_k \, dv \tag{3.2.11}$$

The principle of angular momentum, 3.2.11, will be used, in §3.6, to deduce the symmetry of the Cauchy stress.

3.3 The Cauchy Stress Tensor

3.3.1 The Traction Vector

The **traction vector** was introduced in Part I, §3.3. To recall, it is the limiting value of the ratio of force over area; for Force ΔF acting on a surface element of area ΔS, it is

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \tag{3.3.1}$$

and **n** denotes the normal to the surface element. An infinite number of traction vectors act at a point, each acting on different surfaces through the point, defined by different normals.

3.3.2 Cauchy's Lemma

Cauchy's lemma states that traction vectors acting on opposite sides of a surface are equal and opposite[1]. This can be expressed in vector form:

$$\boxed{\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}} \quad \textbf{Cauchy's Lemma} \tag{3.3.2}$$

This can be proved by applying the principle of linear momentum to a collection of particles of mass Δm instantaneously occupying a small box with parallel surfaces of area Δs, thickness δ and volume $\Delta v = \delta \Delta s$, Fig. 3.3.1. The resultant *surface* force acting on this matter is $\mathbf{t}^{(\mathbf{n})}\Delta s + \mathbf{t}^{(-\mathbf{n})}\Delta s$.

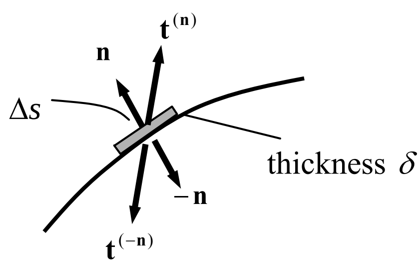


Figure 3.3.1: traction acting on a small portion of material particles

The total linear momentum of the matter is $\int_{\Delta V} \rho \mathbf{v} dv = \int_{\Delta m} \mathbf{v} dm$. By the mean value theorem (see Appendix A to Chapter 1, §1.B.1), this equals $\overline{\mathbf{v}}\Delta m$, where $\overline{\mathbf{v}}$ is the velocity at some interior point. Similarly, the body force acting on the matter is $\int_{\Delta V} \mathbf{b} dv = \overline{\mathbf{b}}\Delta v$, where $\overline{\mathbf{b}}$ is the body force (per unit volume) acting at some interior point. The total mass

[1] this is equivalent to Newton's (third) law of action and reaction – it seems like a lot of work to prove this seemingly obvious result but, to be consistent, it is supposed that the only fundamental dynamic laws available here are the principles of linear and angular momentum, and not any of Newton's laws

can also be written as $\Delta m = \int_{\Delta V} \rho dv = \bar{\rho}\Delta v$. From the principle of linear momentum, Eqn. 3.2.7, and since Δm does not change with time,

$$\mathbf{t}^{(\mathbf{n})}\Delta s + \mathbf{t}^{(-\mathbf{n})}\Delta s + \bar{\mathbf{b}}\Delta v = \frac{d}{dt}\left[\bar{\mathbf{v}}\Delta m\right] = \Delta m \frac{d\bar{\mathbf{v}}}{dt} = \bar{\rho}\Delta v \frac{d\bar{\mathbf{v}}}{dt} = \bar{\rho}\delta\Delta s \frac{d\bar{\mathbf{v}}}{dt} \tag{3.3.3}$$

Dividing through by Δs and taking the limit as $\delta \to 0$, one finds that $\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}$. Note that the values of $\mathbf{t}^{(\mathbf{n})}$, $\mathbf{t}^{(-\mathbf{n})}$ acting on the box with finite thickness are not the same as the final values, but approach the final values *at* the surface as $\delta \to 0$.

3.3.3 Stress

In Part I, the components of the traction vector were called stress components, and it was illustrated how there were nine stress components associated with each material particle. Here, the stress is defined more formally,

Cauchy's Law

Cauchy's Law states that there exists a **Cauchy stress tensor** $\boldsymbol{\sigma}$ which maps the normal to a surface to the traction vector acting on that surface, according to

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \qquad t_i = \sigma_{ij} n_j} \qquad \textbf{Cauchy's Law} \tag{3.3.4}$$

or, in full,

$$\begin{aligned} t_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \\ t_2 &= \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 \\ t_3 &= \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{aligned} \tag{3.3.5}$$

Note:
- many authors define the stress tensor as $\mathbf{t} = \mathbf{n}\boldsymbol{\sigma}$. This amounts to the definition used here since, as mentioned in Part I, and as will be (re-)proved below, the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$, $\sigma_{ij} = \sigma_{ji}$
- the Cauchy stress refers to the *current* configuration, that is, it is a measure of force per unit area acting on a surface in the current configuration.

Stress Components

Taking Cauchy's law to be true (it is proved below), the components of the stress tensor with respect to a Cartesian coordinate system are, from 1.9.4 and 3.3.4,

$$\sigma_{ij} = \mathbf{e}_i \boldsymbol{\sigma} \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{t}^{(\mathbf{e}_j)} \tag{3.3.6}$$

which is the ith component of the traction vector acting on a surface with normal $\mathbf{e}_j$. Note that this definition is inconsistent with that given in Part I, §3.2 – there, the first subscript denoted the direction of the normal – but, again, the two definitions are equivalent because of the symmetry of the stress tensor.

The three traction vectors acting on the surface elements whose outward normals point in the directions of the three base vectors $\mathbf{e}_j$ are

$$\mathbf{t}^{(\mathbf{e}_j)} = \boldsymbol{\sigma}\mathbf{e}_j, \qquad \begin{aligned} \mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \end{aligned} \tag{3.3.7}$$

Eqns. 3.3.6-7 are illustrated in Fig. 3.3.2.

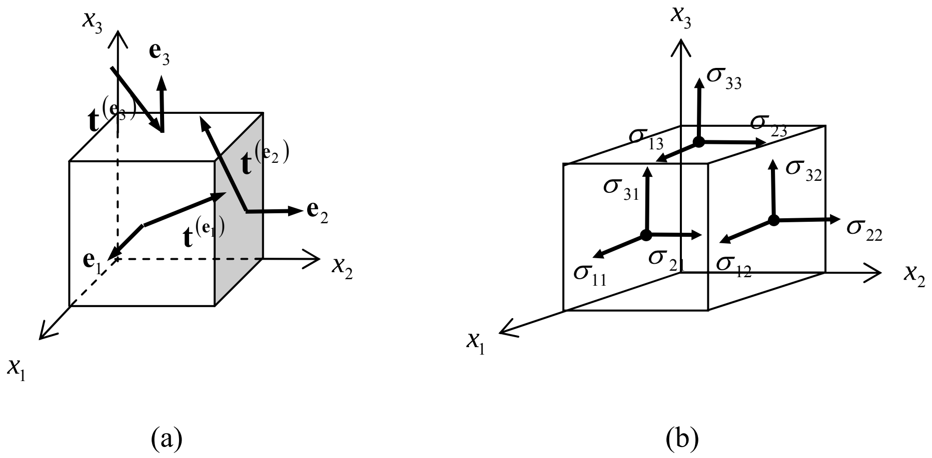


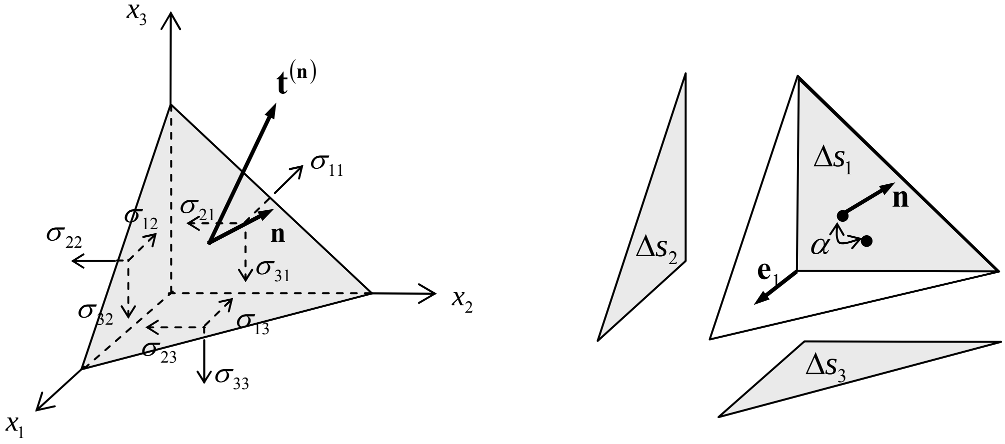
Figure 3.3.2: traction acting on surfaces with normals in the coordinate directions; (a) traction vectors, (b) stress components

Proof of Cauchy's Law

The proof of Cauchy's law essentially follows the same method as used in the proof of Cauchy's lemma.

Consider a small tetrahedral free-body, with vertex at the origin, Fig. 3.3.3. It is required to determine the traction **t** in terms of the nine stress components (which are all shown positive in the diagram).

Let the area of the base of the tetrahedran, with normal **n**, be Δs. The area ds_1 is then $\Delta s \cos\alpha$, where α is the angle between the planes, as shown in Fig. 3.3.3b; this angle is the same as that between the vectors **n** and $\mathbf{e}_1$, so $\Delta s_1 = (\mathbf{n}\cdot\mathbf{e}_1)\Delta s = n_1 \Delta s$, and similarly for the other surfaces: $\Delta s_2 = n_2 \Delta s$ and $\Delta s_3 = n_3 \Delta s$.



(a) (b)

Figure 3.3.3: free body diagram of a tetrahedral portion of material; (a) traction acting on the material, (b) relationship between surface areas and normal components

The resultant surface force on the body, acting in the x_1 direction, is

$$t_1 \Delta s - \sigma_{11} n_1 \Delta s - \sigma_{12} n_2 \Delta s - \sigma_{13} n_3 \Delta s$$

Again, the momentum is $\overline{\mathbf{v}} \Delta M$, the body force is $\overline{\mathbf{b}} \Delta v$ and the mass is $\Delta m = \overline{\rho} \Delta v = \overline{\rho}(h/3)\Delta s$, where *h* is the perpendicular distance from the origin (vertex) to the base. The principle of linear momentum then states that

$$t_1 \Delta s - \sigma_{11} n_1 \Delta s - \sigma_{12} n_2 \Delta s - \sigma_{13} n_3 \Delta s + \overline{b}_1 (h/3)\Delta s = \overline{\rho}(h/3)\Delta s \frac{d\overline{v}_1}{dt}$$

Again, the values of the traction and stress components on the faces will in general vary over the faces, so the values used in this equation are average values over the faces.

Dividing through by Δs, and taking the limit as $h \to 0$, one finds that

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

and now these quantities, t_1, σ_{11}, σ_{12}, σ_{13}, are the values *at* the origin. The equations for the other two traction components can be derived in a similar way.

Normal and Shear Stress

The stress acting normal to a surface is given by

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(\mathbf{n})} \qquad (3.3.8)$$

The shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \tag{3.3.9}$$

Example

The state of stress at a point is given in the matrix form

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine

(a) the traction vector acting on a plane through the point whose unit normal is $\hat{\mathbf{n}} = (1/3)\hat{\mathbf{e}}_1 + (2/3)\hat{\mathbf{e}}_2 - (2/3)\hat{\mathbf{e}}_3$

(b) the component of this traction acting perpendicular to the plane

(c) the shear component of traction.

<u>Solution</u>

(a) The traction is

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

or $\mathbf{t}^{(\hat{\mathbf{n}})} = (-2/3)\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is the projection of $\mathbf{t}^{(\hat{\mathbf{n}})}$ in the direction of $\hat{\mathbf{n}}$, i.e.

$$\sigma_N = \mathbf{t}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\begin{aligned} \sigma_S &= \mathbf{t}^{(\hat{\mathbf{n}})} - (22/9)\hat{\mathbf{n}} \\ &= \left[\left[(-2/3) - (22/27)\right]\hat{\mathbf{e}}_1 + \left[3 - (44/27)\right]\hat{\mathbf{e}}_2 + \left[-1 + (44/27)\right]\hat{\mathbf{e}}_3\right] \\ &= \left[(-40/27)\hat{\mathbf{e}}_1 + (37/27)\hat{\mathbf{e}}_2 + (17/27)\hat{\mathbf{e}}_3\right] \end{aligned}$$

i.e. of magnitude $\sqrt{(-40/27)^2 + (37/27)^2 + (17/27)^2} \approx 2.1$, which equals $\sqrt{\left|\mathbf{t}^{(\hat{\mathbf{n}})}\right|^2 - \sigma_N^2}$.

■

3.4 Properties of the Stress Tensor

3.4.1 Stress Transformation

Let the components of the Cauchy stress tensor in a coordinate system with base vectors $\mathbf{e}_i$ be σ_{ij}. The components in a second coordinate system with base vectors $\mathbf{e}'_j$, σ'_{ij}, are given by the tensor transformation rule 1.10.5:

$$\sigma'_{ij} = Q_{pi} Q_{qj} \sigma_{pq} \tag{3.4.1}$$

where Q_{ij} are the direction cosines, $Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$.

Isotropic State of Stress

Suppose the state of stress in a body is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$$

One finds that the application of the tensor transformation rule yields the very same components no matter what the coordinate system. This is termed an **isotropic** state of stress, or a **spherical** state of stress (see §1.13.3). One example of isotropic stress is the stress arising in fluid at rest, which cannot support shear stress, in which case

$$\boldsymbol{\sigma} = -p\mathbf{I} \tag{3.4.2}$$

where the scalar p is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic** state of stress.

A note on the Transformation Formula

Using the vector transformation rule 1.5.5, the traction and normal transform according to $[\mathbf{t}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{t}]$, $[\mathbf{n}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$. Also, Cauchy's law transforms according to $[\mathbf{t}'] = [\boldsymbol{\sigma}'][\mathbf{n}']$ which can be written as $[\mathbf{Q}^{\mathrm{T}}][\mathbf{t}] = [\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$, so that, pre-multiplying by $[\mathbf{Q}]$, and since $[\mathbf{Q}]$ is orthogonal, $[\mathbf{t}] = \{[\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]\}[\mathbf{n}]$, so $[\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]$, which is the inverse tensor transformation rule 1.13.6a, showing the internal consistency of the theory.

In Part I, Newton's law was applied to a material element to derive the two-dimensional stress transformation equations, Eqn. 3.4.7 of Part I. Cauchy's law was proved in a similar way, using the principle of momentum. In fact, Cauchy's law and the stress transformation equations are equivalent. Given the stress components in one coordinate system, the stress transformation equations give the components in a new coordinate system; particularising this, they give the stress components, and thus the traction vector, acting on new surfaces, oriented in some way with respect to the original axes, which is what Cauchy's law does.

3.4.2 Principal Stresses

Since the stress $\boldsymbol{\sigma}$ is a symmetric tensor, it has three real eigenvalues $\sigma_1, \sigma_2, \sigma_3$, called **principal stresses**, and three corresponding orthonormal eigenvectors called **principal directions**. The eigenvalue problem can be written as

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n} \tag{3.4.3}$$

where $\mathbf{n}$ is a principal direction and σ is a scalar principal stress. Since the traction vector is a multiple of the unit normal, σ is a normal stress component. Thus a principal stress is a stress which acts on a plane of zero shear stress, Fig. 3.4.1.

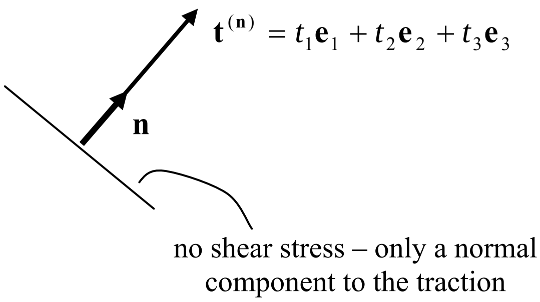


Figure 3.4.1: traction acting on a plane of zero shear stress

The principal stresses are the roots of the characteristic equation 1.11.5,

$$\sigma^3 - \mathbf{I}_1\sigma^2 + \mathbf{I}_2\sigma - \mathbf{I}_3 = 0 \tag{3.4.4}$$

where, Eqn. 1.11.6-7, 1.11.17,

$$\begin{aligned}
I_1 &= \mathrm{tr}\boldsymbol{\sigma} \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \tfrac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\boldsymbol{\sigma}^2\right] \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
&= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\
I_3 &= \tfrac{1}{3}\left[\mathrm{tr}\boldsymbol{\sigma}^3 - \tfrac{3}{2}\mathrm{tr}\boldsymbol{\sigma}\mathrm{tr}\boldsymbol{\sigma}^2 + \tfrac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})^3\right] \\
&= \det\boldsymbol{\sigma} \\
&= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{32} \\
&= \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{3.4.5}$$

The principal stresses and principal directions are properties of the stress tensor, and do not depend on the particular axes chosen to describe the state of stress., and the **stress invariants** I_1, I_2, I_3 are invariant under coordinate transformation. *c.f.* §1.11.1.

If one chooses a coordinate system to coincide with the three eigenvectors, one has the spectral decomposition 1.11.11 and the stress matrix takes the simple form 1.11.12,

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{3.4.6}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions.

3.4.3 Maximum Stresses

Directly from §1.11.3, the three principal stresses include the maximum and minimum normal stress components acting at a point. This result is re-derived here, together with results for the maximum shear stress

Normal Stresses

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions* and consider an arbitrary unit normal vector $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, Fig. 3.4.2. From 3.3.8 and Cauchy's law, the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \tag{3.4.7}$$

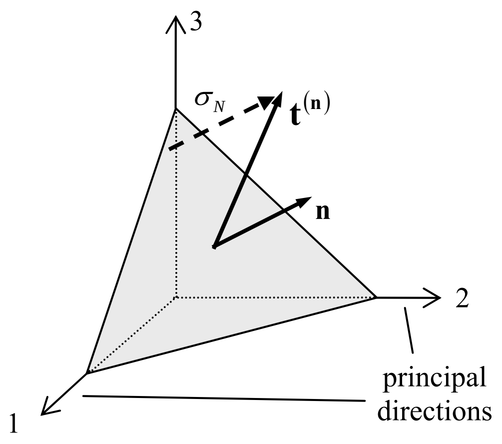


Figure 3.4.2: normal stress acting on a plane defined by the unit normal n

With respect to the principal stresses, using 3.4.6,

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \qquad (3.4.8)$$

and the normal stress is

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \qquad (3.4.9)$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \qquad (3.4.10)$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_3 \qquad (3.4.11)$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses.

From 3.3.39, 3.4.8 and 3.4.9, the shear stress on the plane is

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \qquad (3.4.12)$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right)n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right)n_2^2 + \sigma_3^2 - \left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2 + \sigma_3\right]^2 \qquad (3.4.13)$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned}
\frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1\left(\sigma_1 - \sigma_3\right)\left\{\sigma_1 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0 \\
\frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2\left(\sigma_2 - \sigma_3\right)\left\{\sigma_2 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0
\end{aligned} \qquad (3.4.14)$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 3.4.14 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned} \mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\ \mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\ \mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right| \end{aligned} \tag{3.4.15}$$

Taking $\sigma_1 \ge \sigma_2 \ge \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{3.4.16}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 3.4.3.

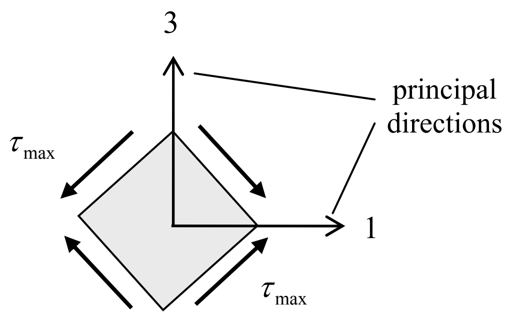


Figure 3.4.3: maximum shear stress at apoint

Example (maximum shear stress)

Consider the stress state

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

This is the same tensor considered in the example of §1.11.1. Using the results of that example, the principal stresses are $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress at that point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{25}{2}$$

The planes and direction upon which they act are shown in Fig. 3.4.4.

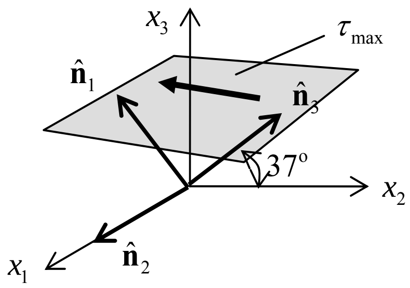


Figure 3.4.4: maximum shear stress

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3.5 Stress Measures for Large Deformations

Thus far, the surface forces acting within a material have been described in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$. The Cauchy stress is also called the **true stress**, to distinguish it from other stress tensors, some of which will be discussed below. It is called the *true* stress because it is a true measure of the force per unit area in the current, deformed, configuration. When the deformations are small, there is no distinction to be made between this deformed configuration and some reference, or undeformed, configuration, and the Cauchy stress is the sensible way of describing the action of surface forces. When the deformations are large, however, one needs to refer to some reference configuration. In this case, there are a number of different possible ways of defining the action of surface forces; some of these stress measures often do not have as clear a physical meaning as the Cauchy stress, but are useful nonetheless.

3.5.1 The First Piola – Kirchhoff Stress Tensor

Consider two configurations of a material, the reference and current configurations. Consider now a vector element of surface in the reference configuration, $\mathbf{N}dS$, where dS is the area of the element and $\mathbf{N}$ is the unit normal. After deformation, the material particles making up this area element now occupy the element defined by $\mathbf{n}ds$, where ds is the area and $\mathbf{n}$ is the normal in the current configuration. Suppose that a force $d\mathbf{f}$ acts on the surface element (in the current configuration). Then by definition of the Cauchy stress

$$d\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}\,ds \tag{3.5.1}$$

The **first Piola-Kirchhoff stress** tensor $\mathbf{P}$ (which will be called the **PK1 stress** for brevity) is defined by

$$d\mathbf{f} = \mathbf{P}\mathbf{N}\,dS \tag{3.5.2}$$

The PK1 stress relates the force acting in the *current* configuration to the surface element in the *reference* configuration. Since it relates to both configurations, it is a two-point tensor.

The (Cauchy) traction vector was defined as

$$\mathbf{t} = \frac{d\mathbf{f}}{ds}, \qquad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \tag{3.5.3}$$

Similarly, one can introduce a **PK1 traction vector T** such that

$$\mathbf{T} = \frac{d\mathbf{f}}{dS}, \qquad \mathbf{T} = \mathbf{P}\mathbf{N} \tag{3.5.4}$$

Whereas the Cauchy traction is the actual physical force per area on the element in the current configuration, the PK1 traction is a fictitious quantity – the force acting on an element in the current configuration divided by the area of the corresponding element in the reference configuration. Note that, since $d\mathbf{f} = \mathbf{t}ds = \mathbf{T}dS$, it follows that $\mathbf{T}$ and $\mathbf{t}$ act in the same direction (but have different magnitudes), Fig. 3.5.1.

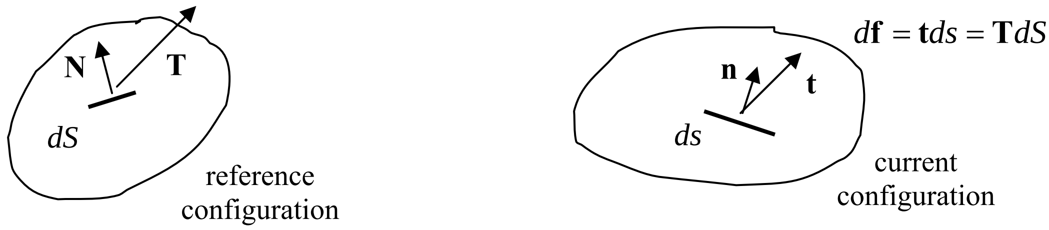

Figure 3.5.1: Traction vectors

Uniaxial Tension

Consider a uniaxial tensile test whereby a specimen is stretched uniformly by a constant force $\mathbf{f}$, Fig. 3.5.2. The initial cross-sectional area of the specimen is A_0 and the cross-sectional area of the specimen at time t is $A(t)$. The Cauchy (true) stress is

$$\boldsymbol{\sigma}(t) = \frac{\mathbf{f}}{A(t)} \qquad (3.5.5)$$

and the PK1 stress is

$$\mathbf{P} = \frac{\mathbf{f}}{A_0} \qquad (3.5.6)$$

This stress measure, force over area of the undeformed specimen, as used in the uniaxial tensile test, is also called the **engineering stress**.

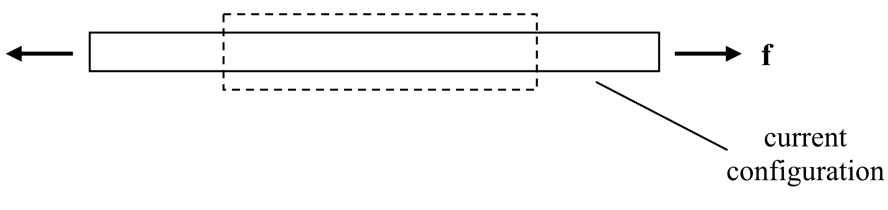

Figure 3.5.2: Uniaxial tension of a bar

The Nominal Stress

The PK1 stress tensor is also called the **nominal stress tensor**. Note that many authors use a different definition for the nominal stress, namely $\mathbf{T} = \mathbf{N}\,\mathbf{P}$, and then define the PK1 stress to be the transpose of this $\mathbf{P}$. Thus all authors use the same definition for the PK1 stress, but a slightly different definition for the nominal stress.

Relation between the Cauchy and PK1 Stresses

From the above definitions,

$$\boldsymbol{\sigma}\,\mathbf{n}ds = \mathbf{P}\,\mathbf{N}dS \qquad (3.5.7)$$

Using Nanson's formula, 2.2.59, $\mathbf{n}ds = J\mathbf{F}^{-\mathrm{T}}\,\mathbf{N}dS$,

$$\boxed{\begin{aligned}\mathbf{P} &= J\boldsymbol{\sigma}\,\mathbf{F}^{-\mathrm{T}}\\ \boldsymbol{\sigma} &= J^{-1}\mathbf{P}\,\mathbf{F}^{\mathrm{T}}\end{aligned}} \qquad \textbf{PK1 stress} \qquad (3.5.8)$$

The Cauchy stress is symmetric, but the deformation gradient is not. Hence the PK1 stress tensor is *not symmetric*, and this restricts its use as an alternative stress measure to the Cauchy stress measure. In fact, this lack of symmetry and lack of a clear physical meaning makes it uncommon for the PK1 stress to be used in the modeling of materials. It is, however, useful in the description of the momentum balance laws in the material description, where **P** plays an analogous role to that played by the Cauchy stress $\boldsymbol{\sigma}$ in the equations of motion (see later).

3.5.2 The Second Piola – Kirchhoff Stress Tensor

The **second Piola – Kirchhoff stress** tensor, or the **PK2 stress**, **S**, is defined by

$$\boxed{\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-\mathrm{T}}} \qquad \textbf{PK2 stress} \qquad (3.5.9)$$

Even though the PK2 does not admit a physical interpretation (except in the simplest of cases, but see the interpretation below), there are three good reasons for using it as a measure of the forces acting in a material. First, one can see that

$$\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)^{\mathrm{T}} = \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$$

and since the Cauchy stress is symmetric, so is the PK2 stress:

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \qquad (3.5.10)$$

A second reason for using the PK2 stress is that, together with the Euler-Lagrange strain **E**, it gives the power of a deforming material (see later). Third, it is parameterized by material coordinates only, that is, it is a material tensor field, in the same way as the Cauchy stress is a spatial tensor field.

Note that the PK1 and PK2 stresses are related through

$$\mathbf{P} = \mathbf{F}\mathbf{S}, \qquad \mathbf{S} = \mathbf{F}^{-1}\mathbf{P} \qquad (3.5.11)$$

The PK2 stress can be interpreted as follows: take the force vector in the current configuration $d\mathbf{f}$ and locate a corresponding vector in the undeformed configuration according to $d\bar{\mathbf{f}} = \mathbf{F}^{-1}d\mathbf{f}$. The PK2 stress tensor is this fictitious force divided by the corresponding area element in the reference configuration: $d\bar{\mathbf{f}} = \mathbf{SN}dS$, and 3.5.9 follows from 3.5.2, 3.5.8:

$$d\mathbf{f} = \mathbf{P}\,\mathbf{N}dS = J\boldsymbol{\sigma}\,\mathbf{F}^{-\mathrm{T}}\,\mathbf{N}dS$$

3.5.3 Alternative Stress Tensors

Some other useful stress measures are described here.

The Kirchhoff Stress

The **Kirchhoff stress tensor** $\boldsymbol{\tau}$ is defined as

$$\boxed{\boldsymbol{\tau} = J\boldsymbol{\sigma}} \qquad \textbf{Kirchhoff Stress} \qquad (3.5.12)$$

It is a spatial tensor field parameterized by spatial coordinates. One reason for its use is that, in many equations, the Cauchy stress appears together with the Jacobian and the use of $\boldsymbol{\tau}$ simplifies formulae.

Note that the Kirchhoff stress is the push forward of the PK2 stress; from 2.12.9b, 2.12.11b,

$$\begin{aligned} \boldsymbol{\tau} &= \chi_*(\mathbf{S})^{\#} = \mathbf{FSF}^{\mathrm{T}} \\ \mathbf{S} &= \chi_*^{-1}(\boldsymbol{\tau})^{\#} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}} \end{aligned} \qquad (3.5.13)$$

The Corotational Cauchy Stress

The **corotational stress** $\hat{\boldsymbol{\sigma}}$ is defined as

$$\boxed{\hat{\boldsymbol{\sigma}} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R}} \qquad \textbf{Corotational Stress} \qquad (3.5.14)$$

where $\mathbf{R}$ is the orthogonal rotation tensor. Whereas the Cauchy stress is related to the PK2 stress through $\boldsymbol{\sigma} = J^{-1}\mathbf{FSF}^{\mathrm{T}}$, the corotational stress is related to the PK2 stress through (with $\mathbf{F}$ replaced by the right (symmetric) stretch tensor $\mathbf{U}$):

$$\hat{\boldsymbol{\sigma}} = J^{-1}\mathbf{USU}^{\mathrm{T}} = J^{-1}\mathbf{U}\left(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)\mathbf{U} = \left(\mathbf{UF}^{-1}\right)\boldsymbol{\sigma}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{U}\right) = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} \qquad (3.5.15)$$

The corotational stress is defined on the intermediate configuration of Fig. 2.10.8. It can be regarded as the push forward of the PK2 stress from the reference configuration through the stretch $\mathbf{U}$, scaled by J^{-1} (Eqn. 2.12.28b):

$$\hat{\boldsymbol{\sigma}} = J^{-1}\chi_*(\mathbf{S})^{\#}{}_{\mathrm{U(G)}} = J^{-1}S^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = J^{-1}S^{ij}\left(\mathbf{UG}_i \otimes \mathbf{UG}_j\right) = J^{-1}\mathbf{USU}^{\mathrm{T}} = J^{-1}\mathbf{USU} \qquad (3.5.16)$$

or as the pull-back of the Cauchy stress with respect to **R** (Eqn. 2.12.27f):

$$\hat{\boldsymbol{\sigma}} = \chi_*^{-1}(\boldsymbol{\sigma})^{\#}{}_{\mathbf{R}(\mathbf{g})} = \sigma^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} \qquad (3.5.17)$$

The Biot Stress

The **Biot** (or **Jaumann**) **stress tensor** $\mathbf{T}_{\mathrm{B}}$ is defined as

$$\boxed{\mathbf{T}_{\mathrm{B}} = \mathbf{R}^{\mathrm{T}}\mathbf{P} = \mathbf{U}\mathbf{S}} \qquad \textbf{Biot Stress} \qquad (3.5.18)$$

From 3.5.11, it is similar to the PK1 stress, only with **F** replaced by **U**.

Example

Consider a **pre-stressed** thin plate with $\sigma_{11} = \sigma_1^0$, $\sigma_{22} = \sigma_2^0$, that is, it has a non-zero stress although no forces are acting[1], Fig. 3.5.3. In this initial state, $\mathbf{F} = \mathbf{I}$ and, considering a two-dimensional state of stress,

$$\boldsymbol{\sigma} = \mathbf{P} = \mathbf{S} = \hat{\boldsymbol{\sigma}} = \boldsymbol{\tau} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix} \sigma_1^0 & 0 \\ 0 & \sigma_2^0 \end{bmatrix}$$

The material is now rotated *as a rigid body* 45° counterclockwise – the stress-state is "frozen" within the material and rotates with it. Then

$$\mathbf{F} = \mathbf{R} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The stress components with respect to the rotated x_i^* axes shown in Fig. 3.5.3b are $\sigma_{11}^* = \sigma_1^0$, etc.; the components with respect to the spatial axes x_i can be found from the stress transformation rule $[\boldsymbol{\sigma}] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}^*][\mathbf{Q}] = [\mathbf{R}][\boldsymbol{\sigma}^*][\mathbf{R}^{\mathrm{T}}]$, and so

$$\boldsymbol{\sigma} = \begin{bmatrix} \frac{1}{2}\left(\sigma_1^0 + \sigma_2^0\right) & \frac{1}{2}\left(\sigma_1^0 - \sigma_2^0\right) \\ \frac{1}{2}\left(\sigma_1^0 - \sigma_2^0\right) & \frac{1}{2}\left(\sigma_1^0 + \sigma_2^0\right) \end{bmatrix}$$

Note that the Cauchy stress changes with this rigid body rotation. Further, with $J = 1$,

$$\boldsymbol{\tau} = \boldsymbol{\sigma}, \quad \mathbf{P} = \begin{bmatrix} \sigma_1^0/\sqrt{2} & -\sigma_2^0/\sqrt{2} \\ \sigma_1^0/\sqrt{2} & \sigma_2^0/\sqrt{2} \end{bmatrix}, \quad \mathbf{S} = \hat{\boldsymbol{\sigma}} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix} \sigma_1^0 & 0 \\ 0 & \sigma_2^0 \end{bmatrix}$$

Note that the PK1 stress is not symmetric. Now attach axes x^* to the material and rotate these axes with the specimen as it rotates, as in Fig. 3.5.3b. The components with respect

[1] for example a piece of metal can be deformed; when the *load is removed* it is often pre-stressed – there is a non-zero state of stress in the material

to these rotated axes give the corotational stress; the corotational stress is the stress in a body, taking out the stress changes caused by rigid body rotations – one says that the corotational stress (and PK2 stress) "rotate" with the body.

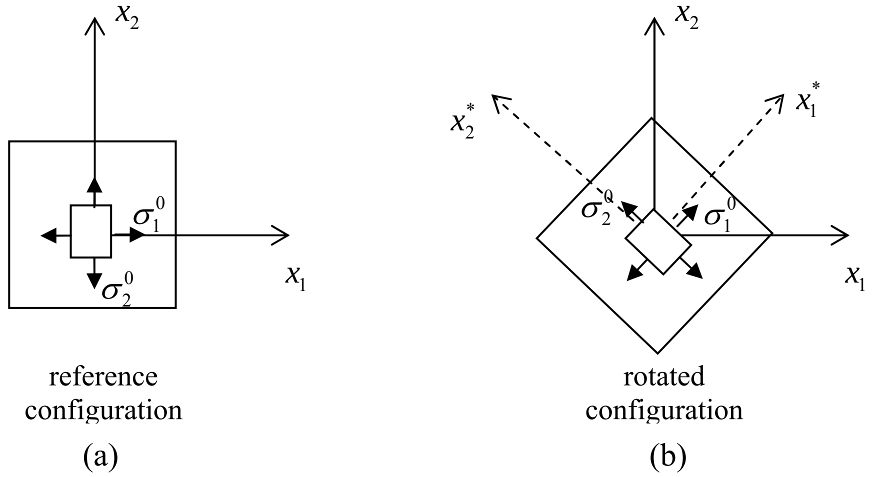

Figure 3.5.3: Pre-stressed material; (a) original position, (b) rotated configuration

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3.5.4 Small deformations

From §2.7, when the deformations are small, neglecting terms involving products of displacement gradients,

$$\mathbf{F} = \mathbf{I} + \mathrm{grad}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2 = \mathbf{I} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \tag{3.5.19}$$

Here, $\mathrm{O}(\mathrm{grad}\mathbf{u})$ means terms of the order of displacement gradients (and higher) have been neglected and $\mathrm{O}(\mathrm{grad}\mathbf{u})^2$ means terms of the order of products of displacement gradients (and higher) have been neglected. Also,

$$\begin{aligned} J &= \det \mathbf{F} \\ &= \det\left(\mathbf{I} + \mathrm{grad}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2\right) = 1 + \mathrm{div}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2 = 1 + \mathrm{O}(\mathrm{grad}\mathbf{u}) \end{aligned} \tag{3.5.20}$$

From 3.5.8 and 3.5.9, using 3.5.19-20, one has

$$\begin{aligned} J\boldsymbol{\sigma} = \mathbf{P}\,\mathbf{F}^{\mathrm{T}} \quad &\rightarrow \quad \boldsymbol{\sigma} + \mathrm{O}(\mathrm{grad}\mathbf{u}) = \mathbf{P} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \\ J\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}\,\mathbf{F}^{\mathrm{T}} \quad &\rightarrow \quad \boldsymbol{\sigma} + \mathrm{O}(\mathrm{grad}\mathbf{u}) = \mathbf{S} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \end{aligned} \tag{3.5.21}$$

In the linear theory then, with $\mathrm{O}(\mathrm{grad}\mathbf{u}) \rightarrow 0$, the stress measures encountered in this section are all equivalent.

3.5.5 Objective Stress Tensors

In order to ascertain the objectivity of the stress tensors, first note that, *by definition*, force is an objective vector, and therefore so also is the traction vector. Similarly for the normal vector. The normal and traction vectors transform under an observer transformation according to 2.8.10, $\mathbf{n}^* = \mathbf{Qn}$ and $\mathbf{t}^* = \mathbf{Qt}$. Then

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \quad \rightarrow \quad \mathbf{Q}^{\mathrm{T}}\mathbf{t}^* = \boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\mathbf{n}^* \quad \rightarrow \quad \mathbf{t}^* = \left(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\right)\mathbf{n}^* \tag{3.5.22}$$

and so $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$; according to 2.8.12, the Cauchy stress is objective. The PK2 stress $\mathbf{S}$ is objective, since it is a material tensor unaffected by an observer transformation. For the PK1 stress, using 2.8.23,

$$\mathbf{P}^* = J^*\boldsymbol{\sigma}^*\left(\mathbf{F}^*\right)^{-\mathrm{T}} = J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\left(\mathbf{QF}\right)^{-\mathrm{T}} = \mathbf{Q}\left(J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right) \tag{3.5.23}$$

and so, according to 2.8.16, $\mathbf{P}$ is objective (transforming like a vector, being a two-point tensor).

3.5.6 Objective Stress Rates

One needs to incorporate stress rates in models of materials where the response depends on the rate of stressing, for example with viscoelastic materials. As discussed in §2.8.5, the rates of objective tensors are not necessarily objective. As discussed in §2.12.3, the Lie derivative of a spatial second order tensor is objective. For the Cauchy stress, there are a number of different objective rates one can use, based on the Lie derivative (see Eqns. 2.8.35-36, 2.12.41, 2.12.44):

$$\begin{array}{lll} \textbf{Cotter-Rivlin stress rate} & \dot{\boldsymbol{\sigma}} + \mathbf{l}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{l} & = \mathrm{L}_{\mathrm{v}}^{\mathrm{b}}\boldsymbol{\sigma} \\ \textbf{Jaumann stress rate} & \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} & = \tfrac{1}{2}\left(\mathrm{L}_{\mathrm{v}}^{\mathrm{b}}\boldsymbol{\sigma} + \mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma}\right) \\ \textbf{Oldroyd stress rate}^2 & \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} & = \mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma} \end{array} \tag{3.5.24}$$

Stress rates of other spatial stress tensors can be defined in the same way, for example the Oldroyd rate of the Kirchhoff stress tensor is $\dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}$.

The material derivative of the material PK2 stress tensor, $\dot{\mathbf{S}}$, is objective. The push forward of $\dot{\mathbf{S}}$ is, from 2.12.9b,

$$\chi_*\left(\dot{\mathbf{S}}\right)^{\#} = \mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} \tag{3.5.25}$$

[2] this is sometimes called the contravariant Oldroyd stress rate, to distinguish it from the Cotter-Rivlin rate, which is also sometimes called the covariant Oldroyd stress rate

This push forward, scaled by the inverse of the Jacobian, $J^{-1}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}}$ is called the **Truesdell stress rate**. This can be expressed in terms of the Cauchy stress by using 3.5.9, and then 2.5.20, 2.5.5:

$$J^{-1}\mathbf{F}\frac{d}{dt}\left(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(\dot{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + J\,\dot{\overline{\mathbf{F}^{-1}}}\,\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\dot{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\boldsymbol{\sigma}\,\dot{\overline{\mathbf{F}^{-\mathrm{T}}}}\right)\mathbf{F}^{\mathrm{T}}$$

$$= \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \operatorname{tr}(\mathbf{d})\boldsymbol{\sigma} \tag{3.5.26}$$

Thus far, objective rates have been constructed by pulling back, taking derivatives and pushing forward. One can construct objective rates also by pulling back and pushing forward with the rotation tensor **R** only, since it is the rotation which causes the stress rates to be non-objective. For example, $\mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma}$, setting $\mathbf{F} = \mathbf{R}$, is, from 3.5.17 and 2.12.27b,

$$\begin{aligned}\chi_*\left(\frac{d}{dt}\left[\chi_*^{-1}(\boldsymbol{\sigma})^{\#}_{\mathbf{R}(\mathbf{g})}\right]\right)_{\mathbf{R}(\hat{\mathbf{g}})} &= \chi_*\left(\frac{d}{dt}\left[\hat{\boldsymbol{\sigma}}\right]\right)^{\#}_{\mathbf{R}(\hat{\mathbf{g}})} \\ &= \mathbf{R}\left(\dot{\mathbf{R}}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\dot{\mathbf{R}}\right)\mathbf{R}^{\mathrm{T}} \\ &= \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\Omega}_R - \boldsymbol{\Omega}_R\boldsymbol{\sigma}\end{aligned} \tag{3.5.27}$$

where $\boldsymbol{\Omega}_R = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ is the skew-symmetric angular velocity tensor 2.6.3. The stress rate 3.5.27 is called the **Green-Naghdi stress rate**. From the above, the Green-Naghdi rate is the push forward of the time derivative of the corotational stress.

Example

Consider again the example discussed at the end of §3.5.3, only let the plate rotate at constant angular velocity ω, so

$$\mathbf{F} = \mathbf{R} = \begin{bmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{bmatrix}, \quad \dot{\mathbf{F}} = \dot{\mathbf{R}} = \omega\begin{bmatrix}-\sin(\omega t) & -\cos(\omega t)\\ \cos(\omega t) & -\sin(\omega t)\end{bmatrix}$$

Again, using the stress transformation rule $[\boldsymbol{\sigma}] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}^*][\mathbf{Q}] = [\mathbf{R}][\boldsymbol{\sigma}^*][\mathbf{R}^{\mathrm{T}}]$,

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma_1^0\cos^2(\omega t) + \sigma_2^0\sin^2(\omega t) & \cos(\omega t)\sin(\omega t)(\sigma_1^0 - \sigma_2^0)\\ \cos(\omega t)\sin(\omega t)(\sigma_1^0 - \sigma_2^0) & \sigma_1^0\sin^2(\omega t) + \sigma_2^0\cos^2(\omega t)\end{bmatrix}$$

and, with $J = 1$,

$$\boldsymbol{\tau} = \boldsymbol{\sigma}, \quad \mathbf{P} = \begin{bmatrix}\cos(\omega t)\sigma_1^0 & -\sin(\omega t)\sigma_2^0\\ \sin(\omega t)\sigma_1^0 & \cos(\omega t)\sigma_2^0\end{bmatrix}, \quad \mathbf{S} = \hat{\boldsymbol{\sigma}} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix}\sigma_1^0 & 0\\ 0 & \sigma_2^0\end{bmatrix}$$

Also,

$$\mathbf{l} = \mathbf{w} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \boldsymbol{\Omega}_{\mathbf{R}} = \omega\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\dot{\mathbf{P}} = \omega\begin{bmatrix} -\sin(\omega t)\sigma_1^0 & -\cos(\omega t)\sigma_2^0 \\ \cos(\omega t)\sigma_1^0 & -\sin(\omega t)\sigma_2^0 \end{bmatrix}, \quad \dot{\mathbf{S}} = \dot{\hat{\boldsymbol{\sigma}}} = \dot{\mathbf{T}}_{\mathrm{B}} = \mathbf{0}$$

and

$$\dot{\boldsymbol{\sigma}} = \omega\begin{bmatrix} -2\sin(\omega t)\cos(\omega t)(\sigma_1^0 - \sigma_2^0) & (\cos^2(\omega t) - \sin^2(\omega t))(\sigma_1^0 - \sigma_2^0) \\ (\cos^2(\omega t) - \sin^2(\omega t))(\sigma_1^0 - \sigma_2^0) & +2\sin(\omega t)\cos(\omega t)(\sigma_1^0 - \sigma_2^0) \end{bmatrix}$$

For a rigid body rotation, it can be seen that the definitions of the Cotter-Rivlin, Jaumann, Oldroyd, Truesdell and Green-Naghdi rates are equivalent, and they are all zero:

$$\dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} = \mathbf{0}$$

This is as expected since objective stress rates for two configurations which differ by a rigid body rotation will, by definition, be equal (the stress components will not change); they are zero in the reference configuration and so will be zero in the rotated configuration.

■

3.5.7 Problems

1. Consider the case of uniaxial stress, where a material with initial dimensions length l_0, breadth w_0 and height h_0 deforms into a component with dimensions length l, breadth w and height h. The only non-zero Cauchy stress component is σ_{11}, acting in the direction of the length of the component.
   (a) write down the motion equations in the material description, $\mathbf{x} = \chi(\mathbf{X})$
   (b) calculate the deformation gradient $\mathbf{F}$ and confirm that $J = \det\mathbf{F}$ is the ratio of the volume in the current configuration to that in the initial configuration
   (c) Calculate the PK1 stress. How is it related to the Cauchy stress for this uniaxial stress-state?
   (d) calculate the PK2 stress
2. A material undergoes the deformation
   $$x_1 = 3X_1 t, \quad x_2 = X_1 t + X_2, \quad x_3 = X_3$$
   The Cauchy stress at a point in the material is
   $$[\boldsymbol{\sigma}] = \begin{bmatrix} t & -2t & 0 \\ -2t & t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
   (a) Calculate the PK1 and PK2 stresses at the point (check that PK2 is symmetric)

(b) Calculate the expressions $\mathbf{P} : \dot{\mathbf{F}}$, $J\boldsymbol{\sigma} : \mathbf{d}$, $\mathbf{S} : \dot{\mathbf{E}}$ (for $\dot{\mathbf{E}}$, use the expression 2.5.18b, $\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$). In these expressions, **d** is the rate of deformation tensor. (You should get the same result for all three cases, since they all give the rate of internal work done by the stresses during the deformation, per unit reference volume – see later)

3. Show that the Oldroyd rate of the Kirchhoff stress, $\dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}$, is equal to the Jacobian times the Truesdell stress rate of the Cauchy stress, 3.5.26.

3.6 The Equations of Motion and Symmetry of Stress

In Part II, §1.1, the Equations of Motion were derived using Newton's Law applied to a differential material element. Here, they are derived using the principle of linear momentum.

3.6.1 The Equations of Motion (Spatial Form)

Application of Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ and the divergence theorem 1.14.21 to 3.2.7 leads directly to the global form of the equations of motion

$$\int_v [\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}]dv = \int_v \rho\,\dot{\mathbf{v}}\,dv, \qquad \int_v \left[\frac{\partial \sigma_{ij}}{\partial x_j} + b_i\right]dv = \int_v \rho\,\dot{v}_i\,dv \tag{3.6.1}$$

The corresponding local form is then

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\frac{d\mathbf{v}}{dt}, \qquad \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho\frac{dv_i}{dt}} \quad \textbf{Equations of Motion} \tag{3.6.2}$$

The term on the right is called the inertial, or kinetic, term, representing the change in momentum. The material time derivative of the spatial velocity field is

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v} \quad \text{so} \quad \frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial x_1}v_1 + \frac{\partial v_1}{\partial x_2}v_2 + \frac{\partial v_1}{\partial x_3}v_3\right), \text{ etc.}$$

and it can be seen that the equations of motion are non-linear in the velocities.

Equations of Equilibrium

When the acceleration is zero, the equations reduce to the equations of equilibrium,

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = 0} \quad \textbf{Equations of Equilibrium} \tag{3.6.3}$$

Flows

A **flow** is a set of quantities associated with the system of forces **t** and **b**, for example the quantities $\mathbf{v}, \boldsymbol{\sigma}, \rho$. A flow is **steady** if the associated spatial quantities are independent of time. A **potential flow** is one for which the velocity field can be written as the gradient of a scalar function, $\mathbf{v} = \operatorname{grad}\phi$. An **irrotational flow** is one for which $\operatorname{curl}\mathbf{v} = 0$.

3.6.2 The Equations of Motion (Material Form)

In the spatial form, the linear momentum of a mass element is $\rho \mathbf{v} dv$. In the material form it is $\rho_0 \mathbf{V} dV$. Here, **V** is the same velocity as **v**, only it is now expressed in terms of the material coordinates **X**, and $\rho dv = \rho_0 dV$. The linear momentum of a collection of material particles occupying the volume v in the current configuration can thus be expressed in terms of an integral over the corresponding volume V in the reference configuration:

$$\boxed{\mathbf{L}(t) = \int_V \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)dV} \qquad \textbf{Linear Momentum (Material Form)} \qquad (3.6.4)$$

and the principle of linear momentum is now, using 3.1.31,

$$\frac{d}{dt}\int_V \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \equiv \mathbf{F}(t) \qquad (3.6.5)$$

The external forces **F** to be considered are those acting on the *current* configuration. Suppose that the surface force acting on a surface element ds in the current configuration is $d\mathbf{f}_{\text{surf}} = \mathbf{t}ds = \mathbf{T}dS$, where **t** and **T** are, respectively, the Cauchy traction vector and the PK1 traction vector (Eqns. 3.5.3-4). Also, just as the PK1 stress measures the actual force in the current configuration, but per unit surface area in the reference configuration, one can introduce the **reference body force B**: this is the actual body force acting in the current configuration, per unit volume in the reference configuration. Thus if the body force acting on a volume element dv in the current configuration is $d\mathbf{f}_{\text{body}}$, then

$$d\mathbf{f}_{\text{body}} = \mathbf{b}dv = \mathbf{B}dV \qquad (3.6.6)$$

The resultant force acting on the body is then

$$\mathbf{F}(t) = \int_S \mathbf{T}dS + \int_V \mathbf{B}dV, \qquad F_i = \int_S T_i\, dS + \int_V B_i\, dV \qquad (3.6.7)$$

Using Cauchy's law, $\mathbf{T} = \mathbf{PN}$, where **P** is the PK1 stress, and the divergence theorem 1.12.21, 3.6.5 and 3.6.7 lead to

$$\int_V [\text{Div}\mathbf{P} + \mathbf{B}]dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \qquad (3.6.8)$$

and the corresponding local form is

$$\boxed{\text{Div}\,\mathbf{P} + \mathbf{B} = \rho_0 \frac{d\mathbf{V}}{dt}, \qquad \frac{\partial P_{ij}}{\partial X_j} + B_i = \rho_0 \frac{dV_i}{dt}}$$

Equations of Motion (Material Form) (3.6.9)

Derivation from the Spatial Form

The equations of motion can also be derived directly from the spatial equations. In order to do this, one must first show that $\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right)$ is zero. One finds that (using the divergence theorem, Nanson's formula 2.2.59 and the fact that $\mathrm{div}\mathbf{I} = 0$)

$$\begin{aligned}
&\int_V \mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right)dV = \int_S J\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS = \int_s \mathbf{n}ds = \int_s \mathbf{I}\mathbf{n}ds = \int_v \mathrm{div}\mathbf{I}\,dv = 0 \\
&\int_V \frac{\partial\left(JF_{ji}^{-1}\right)}{\partial X_j}dV = \int_S JF_{ji}^{-1}N_i dS = \int_s n_i ds = \int_s \delta_{ij}n_j ds = \int_v \frac{\partial \delta_{ij}}{\partial x_i}dv = 0
\end{aligned} \tag{3.6.10}$$

This result is known as the **Piola identity**. Thus, with the PK1 stress related to the Cauchy stress through 3.5.8, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, and using identity 1.14.16c,

$$\begin{aligned}
\mathrm{Div}\mathbf{P} &= \mathrm{Div}\left(\boldsymbol{\sigma}\left(J\mathbf{F}^{-\mathrm{T}}\right)\right) \\
&= \boldsymbol{\sigma}\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right) + \mathrm{Grad}\boldsymbol{\sigma} : \left(J\mathbf{F}^{-\mathrm{T}}\right) \\
&= J\mathrm{Grad}\boldsymbol{\sigma} : \mathbf{F}^{-\mathrm{T}}
\end{aligned} \tag{3.6.11}$$

From 2.2.8c,

$$\mathrm{Div}\mathbf{P} = J\mathrm{div}\boldsymbol{\sigma} \tag{3.6.12}$$

Then, with $dv = JdV$ and 3.6.6, the equations of motion in the spatial form can now be transformed according to

$$\int_v \left[\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}\right]dv = \int_v \rho\,\dot{\mathbf{v}}dv \quad \to \quad \int_V \left[\mathrm{Div}\mathbf{P} + \mathbf{B}\right]dV = \int_V \rho_0 \dot{\mathbf{V}}dV$$

as before.

3.6.3 Symmetry of the Cauchy Stress

It will now be shown that the principle of angular momentum leads to the requirement that the Cauchy stress tensor is symmetric. Applying Cauchy's law to 3.2.11,

$$\begin{aligned}
&\int_s \mathbf{r}\times(\boldsymbol{\sigma}\mathbf{n})ds + \int_v \mathbf{r}\times\mathbf{b}dv = \frac{d}{dt}\int_v \mathbf{r}\times\rho\mathbf{v}dv \\
&\int_s \varepsilon_{ijk}x_j\sigma_{kl}n_l dS + \int_v \varepsilon_{ijk}x_j b_k\, dv = \frac{d}{dt}\int_v \varepsilon_{ijk}x_j\rho v_k\, dv
\end{aligned} \tag{3.6.13}$$

The surface integral can be converted into a volume integral using the divergence theorem. Using the index notation, and concentrating on the integrand of the resulting volume integral, one has, using 1.3.14 (the permutation symbol is a constant here, $\partial\varepsilon_{ijk}/\partial x_l = 0$),

$$\varepsilon_{ijk} \frac{\partial \left(x_j \sigma_{kl} \right)}{\partial x_l} = \varepsilon_{ijk} \left\{ x_j \frac{\partial \sigma_{kl}}{\partial x_l} + \sigma_{kl} \delta_{jl} \right\} = \varepsilon_{ijk} \left\{ x_j \frac{\partial \sigma_{kl}}{\partial x_l} + \sigma_{kj} \right\} \equiv \mathbf{r} \times \operatorname{div} \boldsymbol{\sigma} + \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} \qquad (3.6.14)$$

where E is the third-order permutation tensor, Eqn. 1.9.6, $\mathsf{E} = \varepsilon_{ijk} \left(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \right)$. Thus, with the Reynold's transport identity 3.1.31,

$$\int_v \left\{ \mathbf{r} \times \operatorname{div} \boldsymbol{\sigma} + \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} \right\} dv + \int_v \mathbf{r} \times \mathbf{b} dv = \int_v \rho \frac{d}{dt} \left(\mathbf{r} \times \mathbf{v} \right) dv \qquad (3.6.15)$$

The material derivative of this cross product is

$$\frac{d}{dt} \left(\mathbf{r} \times \mathbf{v} \right) = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{v} = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \mathbf{v} \times \mathbf{v} = \mathbf{r} \times \frac{d\mathbf{v}}{dt} \qquad (3.6.16)$$

and so

$$\int_v \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} dv + \int_v \mathbf{r} \times \left\{ \operatorname{div} \boldsymbol{\sigma} + \mathbf{b} - \rho \frac{d\mathbf{v}}{dt} \right\} dv = 0 \qquad (3.6.17)$$

From the equations of motion 2.6.2, the term inside the brackets is zero, so that

$$\mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} = 0, \qquad \varepsilon_{ijk} \sigma_{kj} = 0 \qquad (3.6.18)$$

It follows, from expansion of this relation, that the matrix of stress components must be symmetric:

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}, \qquad \sigma_{ij} = \sigma_{ji}} \qquad \textbf{Symmetry of Stress} \qquad (3.6.19)$$

3.6.4 Consequences in the Material Form

Here, the consequences of 3.6.19 on the PK1 and PK2 stresses is examined. Using the result $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ and 3.5.8, $\boldsymbol{\sigma} = J^{-1} \mathbf{P} \mathbf{F}^{\mathrm{T}}$,

$$J^{-1} \mathbf{P} \mathbf{F}^{\mathrm{T}} = \left(J^{-1} \mathbf{P} \mathbf{F}^{\mathrm{T}} \right)^{\mathrm{T}} = J^{-1} \mathbf{F} \mathbf{P}^{\mathrm{T}} \qquad (3.6.20)$$

so that

$$\mathbf{P} \mathbf{F}^{\mathrm{T}} = \mathbf{F} \mathbf{P}^{\mathrm{T}}, \qquad P_{ik} F_{jk} = F_{ik} P_{jk} \qquad (3.6.21)$$

These equations are trivial when $i = j$, not providing any constraint on **P**. On the other hand, when $i \neq j$ one has the three equations

$$\begin{aligned}
P_{11}F_{21} + P_{12}F_{22} + P_{13}F_{23} &= F_{11}P_{21} + F_{12}P_{22} + F_{13}P_{23} \\
P_{11}F_{31} + P_{12}F_{32} + P_{13}F_{33} &= F_{11}P_{31} + F_{12}P_{32} + F_{13}P_{33} \\
P_{21}F_{31} + P_{22}F_{32} + P_{23}F_{33} &= F_{21}P_{31} + F_{22}P_{32} + F_{23}P_{33}
\end{aligned} \tag{3.6.22}$$

Thus angular momentum considerations imposes these three constraints on the PK1 stress (as they imposed the three constraints $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$ on the Cauchy stress).

It has already been seen that a consequence of the symmetry of the Cauchy stress is the symmetry of the PK2 stress **S**; thus, formally, the symmetry of **S** is the result of the angular momentum principle.

3.7 Boundary Conditions and The Boundary Value Problem

In order to solve a mechanics problem, one must specify certain conditions around the boundary of the material under consideration. Such **boundary conditions** will be discussed here, together with the resulting **boundary value problem** (**BVP**). (see Part I, 3.5.1, for a discussion of stress boundary conditions.)

3.7.1 Boundary Conditions

There are two types of boundary condition, those on displacement and those on traction. Denote the body in the reference condition by B_0 and in the current configuration by *B*. Denote the boundary of the body in the reference configuration by S and in the current configuration by *s*, Fig. 3.7.1.

Displacement Boundary Conditions

The position of particles may be specified over some portion of the boundary in the current configuration. That is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is specified to be $\bar{\mathbf{x}}$ say, over some portion $s_\mathbf{u}$ of *s*, Fig. 3.7.1, which corresponds to the portion $S_\mathbf{u}$ of S. With $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{X}(\mathbf{x})$, or $\mathbf{U}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$, this can be expressed as

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \bar{\mathbf{u}}(\mathbf{x}), \quad && \mathbf{x} \in s_\mathbf{u} \\ \mathbf{U}(\mathbf{X}) &= \bar{\mathbf{U}}(\mathbf{X}), \quad && \mathbf{X} \in S_\mathbf{u} \end{aligned} \tag{3.7.1}$$

These are called **displacement boundary conditions**. The most commonly encountered displacement boundary condition is where some portion of the boundary is fixed, in which case $\bar{\mathbf{u}}(\mathbf{x}) = \mathbf{o}$.

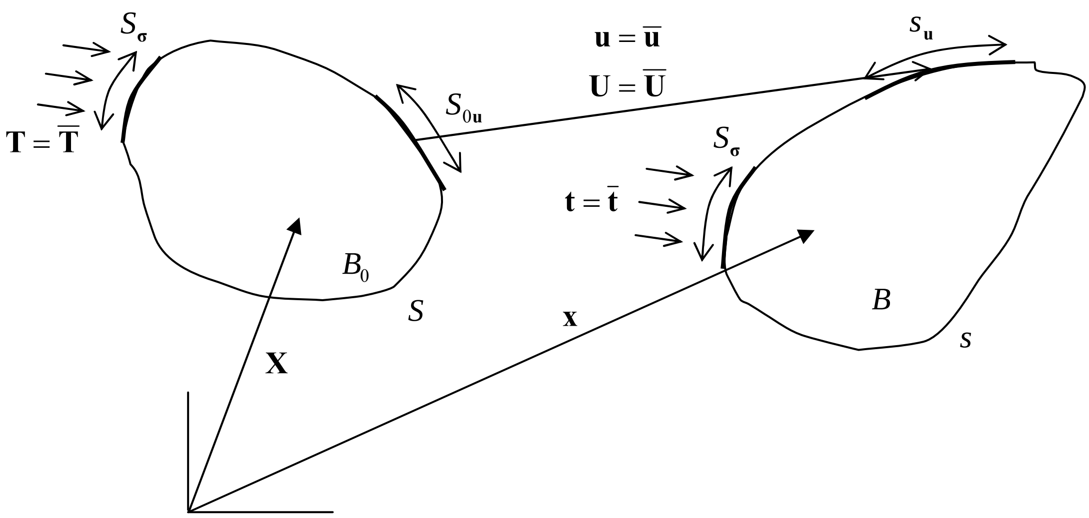

Figure 3.7.1: Boundary conditions

Traction Boundary Conditions

Traction $\mathbf{t} = \bar{\mathbf{t}}$ can be specified over a portion $s_{\boldsymbol{\sigma}}$ of the boundary, Fig. 3.7.1. These traction boundary conditions are related to the PK1 traction $\mathbf{T} = \overline{\mathbf{T}}$ over the corresponding surface $S_{\boldsymbol{\sigma}}$ in the reference configuration, through Eqns. 3.5.1-4,

$$\mathbf{T}dS = \mathbf{PN}dS = \mathbf{t}ds = \boldsymbol{\sigma}\mathbf{n}ds \tag{3.7.2}$$

One usually knows the position of the boundary S and the normal $\mathbf{N}(\mathbf{X})$ in the reference configuration. As deformation proceeds, the PK1 traction develops according to $\overline{\mathbf{T}} = \mathbf{PN}$ with, from 3.5.8, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$. The PK1 stress will in general depend on the motion $\mathbf{x}$ and the deformation gradient $\mathbf{F}$, so the traction boundary condition can be expressed in the general form

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}, \mathbf{x}, \mathbf{F}) \tag{3.7.3}$$

Example: Fluid Pressure

Consider the case of fluid pressure p around the boundary, $\bar{\mathbf{t}} = -p\mathbf{n}$, Fig. 3.7.2. The Cauchy traction $\bar{\mathbf{t}}$ depends through the normal $\mathbf{n}$ on the new position and geometry of the surface $s_{\boldsymbol{\sigma}}$. Also, $\overline{\mathbf{T}} = -pJ\mathbf{F}^{-\mathrm{T}}\mathbf{N}$, which is of the general form 3.7.3.

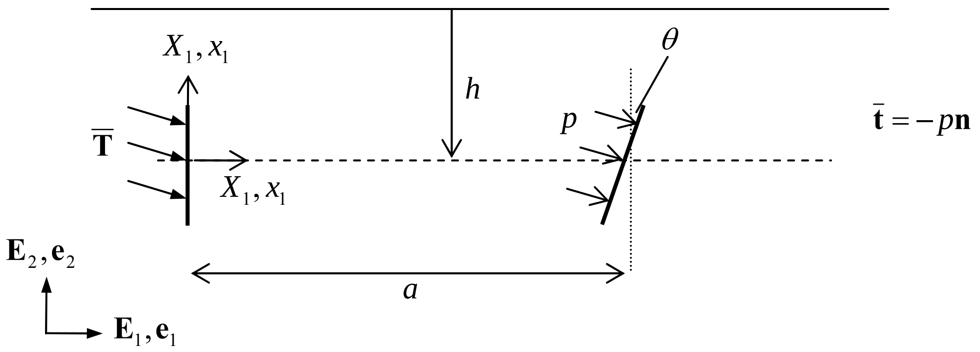

Figure 3.7.2: Fluid pressure on deforming material

Consider a material under water with part of its surface deforming as shown in Fig. 3.7.2. Referring to the figure, $\mathbf{N} = -\mathbf{E}_1$, $\mathbf{n} = -\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, $\boldsymbol{\sigma} = -p\mathbf{I}$, $p = \rho g(h - x_2)$ and

$$\begin{aligned} x_1 &= X_1 + a + X_2 \tan\theta \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned}, \quad \mathbf{F} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \det\mathbf{F} = 1$$

The traction vectors and PK1 stress are

$$\bar{\mathbf{t}} = -\rho g(h - x_2)\begin{bmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \overline{\mathbf{T}} = -\rho g(h - X_2)\begin{bmatrix} -1 \\ \tan\theta \\ 0 \end{bmatrix}, \quad \mathbf{P} = -\rho g(h - x_2)\begin{bmatrix} 1 & 0 & 0 \\ -\tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with (note that $dS / ds = \cos\theta$) $|\bar{\mathbf{t}}| = p$ and $|\overline{\mathbf{T}}| = p / \cos\theta$. The traction vectors clearly depend on both position, and the deformation through θ. In this example, $\text{grad}\mathbf{u} = \mathbf{F} - \mathbf{I} = \text{Grad}\mathbf{U} = \mathbf{I} - \mathbf{F}^{-1} = \tan\theta \mathbf{e}_1 \otimes \mathbf{e}_2$ and

$$\theta(\text{grad}\mathbf{u}) = \arctan\|\text{grad}\mathbf{u}\| = \arctan\sqrt{\text{grad}\mathbf{u} : \text{grad}\mathbf{u}}$$

■

Dead Loading

A special case of loading is that of **dead loading**, where

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}) \tag{3.7.4}$$

Here, the PK1 stress on the boundary does not change with the deformation and an initially normal traction will not remain so as deformation proceeds.

For example, if one considers again the geometry of Fig. 3.7.2, this time take

$$\overline{\mathbf{T}}(\mathbf{X}) = \mathbf{PN} = -p\mathbf{N} = \rho g(h - X_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}(\mathbf{X}) = -\rho g(h - X_2)\mathbf{I}$$

Then

$$\bar{\mathbf{t}}(\mathbf{x}, \theta) = \cos\theta \rho g(h - x_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}(\mathbf{x}, \theta) = -\rho g(h - x_2)\begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.7.2 The Boundary Value Problem

The equations of motion 3.6.2, 3.6.9, are a set of three differential equations. In the solution of any problem, one would have to supplement these equations with others, for example a constitutive equation expressing a relationship between the stress and the kinematic variables (see Part IV). This constitutive relation will typically relate the stress to the strains, or rates of strain, for example $\boldsymbol{\sigma} = f(\mathbf{e}, \mathbf{d})$. Suppose then that the stresses are known in terms of the strains and hence the displacements $\mathbf{u}$. The equations of motion are then a set of three second order differential equations in the three unknowns u_i (assuming that the body force $\mathbf{b}$ is a prescribed function of the problem). They need to be subjected to certain boundary and initial conditions.

Assume that the boundary conditions are such that the displacements are specified over that part of the surface $s_{\mathbf{u}}$ and tractions are specified over that part $s_{\boldsymbol{\sigma}}$, with the total surface $s = s_{\mathbf{u}} + s_{\boldsymbol{\sigma}}$, with $s_{\mathbf{u}} \cap s_{\boldsymbol{\sigma}} = 0$ [1]. Thus

$$\begin{aligned} \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, &\quad \text{on } s_\sigma \\ \mathbf{u} = \bar{\mathbf{u}}, &\quad \text{on } s_u \end{aligned} \qquad \textbf{Boundary Conditions} \qquad (3.7.5)$$

where the overbar signifies quantities which are prescribed. Initial conditions are also required for the displacement and velocity, so that

$$\begin{aligned} \mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x}), &\quad \text{at } t = 0 \\ \dot{\mathbf{u}}(\mathbf{x},\mathbf{t}) = \dot{\mathbf{u}}_0(\mathbf{x}), &\quad \text{at } t = 0 \end{aligned} \qquad \textbf{Initial Conditions} \qquad (3.7.6)$$

and it is usually taken that $\mathbf{x} = \mathbf{X}$ at $t = 0$. Comparing 3.7.5 and 3.7.6, one also requires that $\mathbf{u}_0 = \bar{\mathbf{u}}$, $\dot{\mathbf{u}}_0 = \dot{\bar{\mathbf{u}}}$ over $s_{\mathbf{u}}$, so that the boundary and initial conditions are compatible.

These equations together, the differential equations of motion and the boundary and initial conditions, are called the **strong form** of the initial boundary value problem (BVP):

$$\boxed{\begin{aligned} &\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\dot{\mathbf{v}} = \rho\ddot{\mathbf{u}} \\ &\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, \quad \text{on } s_\sigma \\ &\mathbf{u} = \bar{\mathbf{u}}, \quad \text{on } s_u \\ &\mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x}), \quad \text{at } t = 0 \\ &\dot{\mathbf{u}}(\mathbf{x},\mathbf{t}) = \dot{\mathbf{u}}_0(\mathbf{x}), \quad \text{at } t = 0 \end{aligned}} \qquad \textbf{Strong form of the Initial BVP} \qquad (3.7.7)$$

When the problem is quasi-static, so the accelerations can be neglected, the equations of motion reduce to the equations of equilibrium 3.6.3. In that case one does not need initial conditions and one has a boundary value problem involving 3.7.5 only.

It is only in certain special cases and in certain simple problems that an exact solution can be obtained to these equations. An alternative solution strategy is to convert these equations into what is known as the **weak form**. The weak form, which is in the form of integrals rather than differential equations, can then be solved approximately using a numerical technique, for example the Finite Element Method[2]. The weak form is discussed in §3.9.

---

[1] It is possible to specify both traction and displacement over the same portion of the boundary, but not the same components. For example, if one specified $\mathbf{t} = t_1\mathbf{e}_1$ on a boundary, one could also specify $\mathbf{u} = u_2\mathbf{e}_2$, but not $\mathbf{u} = u_1\mathbf{e}_1$. In that case, one could imagine the boundary to consist of two separate boundaries, one with conditions with respect to $\mathbf{e}_1$ and one with respect to $\mathbf{e}_2$, and still write $s_{\mathbf{u}} \cap s_{\boldsymbol{\sigma}} = 0$.

[2] Further, it is often easier to prove results regarding the uniqueness and stability of solutions to the problem when it is cast in the weak form

In the material form, the boundary conditions are

$$\begin{aligned} &\mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}}, && \text{on } S_\sigma \\ &\mathbf{U} = \overline{\mathbf{U}}, && \text{on } S_u \end{aligned} \qquad \textbf{Boundary Conditions} \qquad (3.7.8)$$

and the initial conditions are

$$\begin{aligned} &\mathbf{U}(\mathbf{X},\mathbf{t}) = \mathbf{U}_0(\mathbf{X}), && \text{at } t = 0 \\ &\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) = \dot{\mathbf{U}}_0(\mathbf{X}), && \text{at } t = 0 \end{aligned} \qquad \textbf{Initial Conditions} \qquad (3.7.9)$$

and the initial vale problem is

$$\boxed{\begin{aligned} &\mathrm{Div}\mathbf{P} + \mathbf{B} = \rho_0 \dot{\mathbf{V}} = \rho \ddot{\mathbf{U}} \\ &\mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}}, && \text{on } S_\sigma \\ &\mathbf{U} = \overline{\mathbf{U}}, && \text{on } S_u \\ &\mathbf{U}(\mathbf{X},\mathbf{t}) = \mathbf{U}_0(\mathbf{X}), && \text{at } t = 0 \\ &\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) = \dot{\mathbf{U}}_0(\mathbf{X}), && \text{at } t = 0 \end{aligned}} \qquad \textbf{Strong form of the Initial BVP} \qquad (3.7.10)$$

3.8 Balance of Mechanical Energy

3.8.1 The Balance of Mechanical Energy

First, from Part I, Chapter 5, recall work and kinetic energy are related through

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \qquad (3.8.1)$$

where W_{ext} is the work of the external forces and W_{int} is the work of the internal forces. The *rate* form is

$$P_{\text{ext}} + P_{\text{int}} = \dot{K} \qquad (3.8.2)$$

where the external and internal *powers* and rate of change of kinetic energy are

$$P_{\text{ext}} = \frac{d}{dt} W_{\text{ext}}, \quad P_{\text{int}} = \frac{d}{dt} W_{\text{int}}, \quad \dot{K} = \frac{d}{dt} \Delta K \qquad (3.8.3)$$

This expresses the *mechanical* energy balance for a material. Eqn. 3.8.2 is equivalent to the equations of motion (see below).

The total external force acting on the material is given by 3.2.6:

$$\mathbf{F}_{\text{ext}} = \int_s \mathbf{t}\, ds + \int_v \mathbf{b}\, dv \qquad (3.8.4)$$

The increment in work done dW when an element subjected to a body force (per unit volume) **b** undergoes a displacement $d\mathbf{u}$ is $\mathbf{b} \cdot d\mathbf{u}\, dv$. The rate of working is $dP = \mathbf{b} \cdot (d\mathbf{u} / dt) dv$. Thus, and similarly for the traction, the power of the external forces is

$$P_{\text{ext}} = \int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv \qquad (3.8.5)$$

where **v** is the velocity. Also, the total kinetic energy of the matter in the volume is

$$K = \int_v \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv \qquad (3.8.6)$$

Using Reynold's transport theorem,

$$\frac{d}{dt} K = \int_v \tfrac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dv = \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv \qquad (3.8.7)$$

Thus the expression 3.8.2 becomes

$$\int_s \mathbf{t}\cdot\mathbf{v}\,ds + \int_v \mathbf{b}\cdot\mathbf{v}\,dv + P_{\text{int}} = \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}dv \tag{3.8.8}$$

power of surface forces — power of body forces — power of internal forces — rate of change of kinetic energy

It can be seen that some of the power exerted by the external forces alters the kinetic energy of the material and the remainder changes its internal energy state.

Conservative Force System

In the special case where the internal forces are conservative, that is, no energy is dissipated as heat, but all energy is stored as internal energy, one can express the power of the internal forces in terms of a potential function u (see Part I, §5.1), and rewrite this equation as

$$\int_s \mathbf{t}\cdot\mathbf{v}\,ds + \int_v \mathbf{b}\cdot\mathbf{v}\,dv = \int_v \rho\frac{du}{dt}dv + \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}dv \tag{3.8.9}$$

rate of change of internal energy

Here, the rate of change of the internal energy has been written in the form

$$\frac{d}{dt}U = \frac{d}{dt}\int_v \rho u dv = \int_v \rho\frac{du}{dt}dv \tag{3.8.10}$$

where u is the internal energy per unit mass, or the **specific internal energy**.

3.8.2 The Stress Power

To express the power of the internal forces P_{int} in terms of stresses and strain-rates, first re-write the rate of change of kinetic energy using the equations of motion,

$$\frac{d}{dt}K = \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}dv = \int_v \mathbf{v}\cdot(\operatorname{div}\boldsymbol{\sigma} + \mathbf{b})dv \tag{3.8.11}$$

Also, using the product rule of differentiation,

$$\mathbf{v}\cdot\operatorname{div}\boldsymbol{\sigma} = \operatorname{div}(\mathbf{v}\boldsymbol{\sigma}) - \boldsymbol{\sigma}:\mathbf{l}, \qquad v_i\frac{\partial\sigma_{ij}}{\partial x_j} = \frac{\partial(v_i\sigma_{ij})}{\partial x_j} - \sigma_{ij}\frac{\partial v_i}{\partial x_j} \tag{3.8.12}$$

where $\mathbf{l}$ is the spatial velocity gradient, $l_{ij} = \partial v_i / \partial x_j$. Decomposing $\mathbf{l}$ into its symmetric part $\mathbf{d}$, the rate of deformation, and its antisymmetric part $\mathbf{w}$, the spin tensor, gives

$$\boldsymbol{\sigma} : \mathbf{l} = \boldsymbol{\sigma} : \mathbf{d} + \boldsymbol{\sigma} : \mathbf{w} = \boldsymbol{\sigma} : \mathbf{d}, \qquad \sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{3.8.13}$$

since the double contraction of any symmetric tensor ($\boldsymbol{\sigma}$) with any skew-symmetric tensor ($\mathbf{w}$) is zero, 1.10.31c. Also, using Cauchy's law and the divergence theorem 1.14.21,

$$\begin{gathered} \int_s \mathbf{t} \cdot \mathbf{v}\, ds = \int_s \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{v}\, ds = \int_s (\mathbf{v}\boldsymbol{\sigma}) \cdot \mathbf{n}\, ds = \int_v \operatorname{div}(\mathbf{v}\boldsymbol{\sigma}) dv \\ \int_s t_i v_i\, ds = \int_s \sigma_{ik} n_k v_i\, ds = \int_v \frac{\partial(\sigma_{ik} v_i)}{\partial x_k} dv \end{gathered} \tag{3.8.14}$$

Thus, finally, from Eqn. 3.8.8,

$$P_{\text{int}} = -\int_v \boldsymbol{\sigma} : \mathbf{d}\, dv, \qquad P_{\text{int}} = -\int_v \sigma_{ij} \left\{ \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} dv \qquad \textbf{Stress Power} \tag{3.8.15}$$

The term $\boldsymbol{\sigma} : \mathbf{d}$ is called the **stress power**; the stress power is the (negative of the) rate of working of the internal forces, per unit volume. The complete equation for the conservation of mechanical energy is then

$$\int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv = \int_v \boldsymbol{\sigma} : \mathbf{d}\, dv + \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv \qquad \textbf{Mechanical Energy Balance} \tag{3.8.16}$$

The stress power is that part of the externally supplied power which is not converted into kinetic energy; it is converted into heat and a change in internal energy.

Note that, as with the law of conservation of mechanical energy for a particle, this equation does not express a separate law of continuum mechanics; it is merely a re-arrangement of the equations of motion (see below), which themselves follows from the principle of linear momentum (Newtons second law).

Conservative Force System

If the internal forces are conservative, one has

$$\int_v \boldsymbol{\sigma} : \mathbf{d}\, dv = \frac{d}{dt} U = \int_v \rho \frac{du}{dt} dv \tag{3.8.17}$$

or, in local form,

$$\boxed{\boldsymbol{\sigma} : \mathbf{d} = \rho \frac{du}{dt}} \quad \textbf{Mechanical Energy Balance (Conservative System)} \quad (3.8.18)$$

This is the local form of the energy equation for the case of a purely mechanical conservative process.

3.8.3 Derivation from the Equations of Motion

As mentioned, the conservation of mechanical energy equation can be derived directly from the equations of motion. The derivation is similar to that used above (where the mechanical energy equations were used to derive an expression for the stress power using the equations of motion). One has, multiplying the equations of motion by **v** and integrating,

$$\begin{aligned}
\int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv = \int_v \mathbf{v} \cdot (\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}) dv &= \int_v \{\operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \mathbf{l} + \mathbf{v} \cdot \mathbf{b}\} dv \\
&= \int_v \{\operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \mathbf{d} + \mathbf{v} \cdot \mathbf{b}\} dv \qquad (3.8.19) \\
&= \int_v -\boldsymbol{\sigma} : \mathbf{d} dv + \int_s \mathbf{t} \cdot \mathbf{v} ds + \int_v \mathbf{v} \cdot \mathbf{b} dv
\end{aligned}$$

3.8.4 Stress Power and the Continuum Element

In the above, the stress power was derived using a global (integral) form of the equations. The stress power can also be deduced by considering a differential mass element. For example, consider such an element whose boundary particles are moving with velocity **v** and whose boundary is subjected to stresses $\boldsymbol{\sigma}$, Fig. 3.8.1.

Consider first the components of force and velocity acting in the x_1 direction. The external forces act on the six sides. On three of them (the ones that can be seen in the illustration) the stress and velocity act in the same direction, so the power is positive; on the other three they act in opposite directions, so there the power is negative.

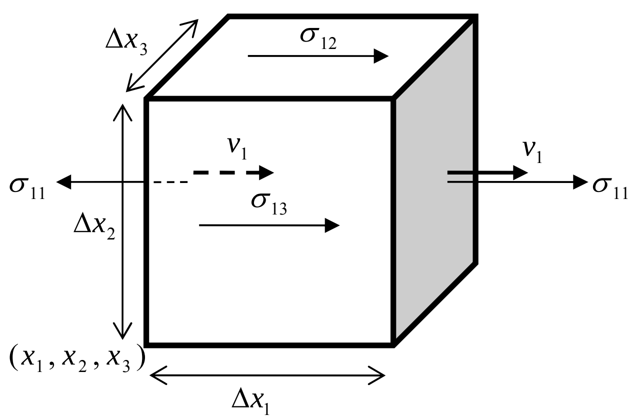

Figure 3.8.1: A differential mass element subjected to stresses

As usual (see §1.6.6), the element is assumed to be small enough so that the product of stress and velocity varies linearly over the element, so that the average of this product over an element face can be taken to be representative of the power of the surface forces on that element. The power of the external surface forces acting on the three faces to the front is then

$$\begin{aligned}P_{\text{surf}} = \Delta x_2 \Delta x_3 (\sigma_{11} v_1)_{x_1+\Delta x_1, x_2+\frac{1}{2}\Delta x_2, x_3+\frac{1}{2}\Delta x_3} + \Delta x_1 \Delta x_3 (\sigma_{12} v_1)_{x_1+\frac{1}{2}\Delta x_1, x_2+\Delta x_2, x_3+\frac{1}{2}\Delta x_3} \\ + \Delta x_1 \Delta x_2 (\sigma_{13} v_1)_{x_1+\frac{1}{2}\Delta x_1, x_2+\frac{1}{2}\Delta x_2, x_3+\Delta x_3}\end{aligned} \tag{3.8.20}$$

Using a Taylor's series expansion, and neglecting higher order terms, then leads to

$$\begin{aligned}P_{\text{surf}} &\approx \Delta x_2 \Delta x_3 \left\{ (\sigma_{11} v_1)_{x_1,x_2,x_3} + \Delta x_1 \frac{\partial(\sigma_{11} v_1)}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial(\sigma_{11} v_1)}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial(\sigma_{11} v_1)}{\partial x_3} \right\} \\ &+ \Delta x_1 \Delta x_3 \left\{ (\sigma_{12} v_1)_{x_1,x_2,x_3} + \tfrac{1}{2}\Delta x_1 \frac{\partial(\sigma_{12} v_1)}{\partial x_1} + \Delta x_2 \frac{\partial(\sigma_{12} v_1)}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial(\sigma_{12} v_1)}{\partial x_3} \right\} \\ &+ \Delta x_1 \Delta x_2 \left\{ (\sigma_{13} v_1)_{x_1,x_2,x_3} + \tfrac{1}{2}\Delta x_1 \frac{\partial(\sigma_{13} v_1)}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial(\sigma_{13} v_1)}{\partial x_2} + \Delta x_3 \frac{\partial(\sigma_{13} v_1)}{\partial x_3} \right\}\end{aligned} \tag{3.8.21}$$

The net power *per unit volume* (subtracting the power of the stresses on the other three surfaces and dividing through by the volume) is then

$$P_{\text{surf}} = \frac{\partial(\sigma_{11} v_1)}{\partial x_1} + \frac{\partial(\sigma_{12} v_1)}{\partial x_2} + \frac{\partial(\sigma_{13} v_1)}{\partial x_3} = \frac{\partial(\sigma_{1j} v_1)}{\partial x_j} \tag{3.8.22}$$

Assume the body force **b** to act at the centre of the element. Neglecting higher order terms which vanish as the element size is allowed to shrink towards zero, the power of the body force in the x_1 direction, per unit volume, is simply $b_1 v_1$.

The total power of the external forces is then (including the other two components of stress and velocity), using the equations of motion,

$$\begin{aligned}P_{\text{ext}} &= \frac{\partial(\sigma_{ij} v_i)}{\partial x_j} + b_i v_i \\ &= \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} v_i + b_i v_i \\ &= \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \left\{ -b_i + \rho \frac{dv_i}{dt} \right\} v_i + b_i v_i \\ &= \sigma_{ij} \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \rho \frac{1}{2} \frac{d(v_i v_i)}{dt}\end{aligned} \qquad \begin{aligned}P_{\text{ext}} &= \operatorname{div}(\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{v}) + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{l} + \operatorname{div}\boldsymbol{\sigma} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{l} + \left\{ -\mathbf{b} + \rho \frac{d\mathbf{v}}{dt} \right\} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{d} + \rho \frac{1}{2} \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt}\end{aligned} \tag{3.8.23}$$

which again equals the stress power term plus the change in kinetic energy.

The power of the internal forces is $-\boldsymbol{\sigma}:\mathbf{d}$, a result of the forces acting *inside* the differential element, *reacting* to the applied forces $\boldsymbol{\sigma}$ and $\mathbf{b}$.

3.8.5 The Balance of Mechanical Energy (Material form)

The material form of the power of the external forces is written as a function of the PK1 traction **T** and the reference body force **B**, 3.6.7, and the kinetic energy as a function of the velocity $\mathbf{V}(\mathbf{X})$:

$$\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV + P_{\text{int}} = \int_V \rho_0\mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV \tag{3.8.24}$$

Next, using the identities 2.5.4, $\dot{\mathbf{F}} = \mathbf{lF}$ and 1.10.3h, $\mathbf{A}:(\mathbf{BC}) = \left(\mathbf{AC}^{\mathrm{T}}\right):\mathbf{B}$, gives

$$\boldsymbol{\sigma}:\mathbf{d} = \boldsymbol{\sigma}:\mathbf{l} - \boldsymbol{\sigma}:\mathbf{w} = \boldsymbol{\sigma}:\mathbf{l} = \boldsymbol{\sigma}:\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}, \tag{3.8.25}$$

and so

$$\begin{aligned}\int_v \boldsymbol{\sigma}:\mathbf{d}\,dv &= \int_v \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}\,dv = \int_V \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}\,J\,dV \\ &= \int_V \mathbf{P}:\dot{\mathbf{F}}\,dV\end{aligned} \tag{3.8.26}$$

and

$$\boxed{\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV = \int_V \mathbf{P}:\dot{\mathbf{F}}\,dV + \int_V \rho_0\mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV}$$

Mechanical Energy Balance (Material Form) (3.8.27)

For a conservative system, this can be written in terms of the internal energy

$$\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV = \int_V \rho_0\frac{du}{dt}dV + \int_V \rho_0\mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV \tag{3.8.28}$$

3.8.6 Work Conjugate Variables

Since the stress power is the double contraction of the Cauchy stress and rate-of-deformation, one says that the Cauchy stress and rate of deformation are **work conjugate** (or **power conjugate** or **energy conjugate**). Similarly, from 3.8.26, the PK1 stress **P** is power conjugate to $\dot{\mathbf{F}}$. It can also be shown that the PK2 stress **S** is power conjugate to the rate of Euler-Lagrange strain, $\dot{\mathbf{E}}$ (and hence also the right Cauchy-Green strain) {▲Problem 1} :

$$J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} : \tfrac{1}{2}\dot{\mathbf{C}} \tag{3.8.29}$$

Note that, for conservative systems, these quantities represent the rate of change of internal energy per unit *reference* volume.

Using the polar decomposition and the relation $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$,

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} \\ &= \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} \\ &= \boldsymbol{\Omega}_{\mathbf{R}}\mathbf{F} + \mathbf{R}\dot{\mathbf{U}} \end{aligned} \tag{3.8.30}$$

where $\boldsymbol{\Omega}_{\mathbf{R}}$ is the angular velocity tensor 2.6.1. Then, using 1.11.3h, 1.10.31c, and the definitions 3.5.8, 3.5.12 and 3.5.18,

$$\begin{aligned} \mathbf{P} : \dot{\mathbf{F}} &= \mathbf{P} : \boldsymbol{\Omega}_{\mathbf{R}}\mathbf{F} + \mathbf{P} : \mathbf{R}\dot{\mathbf{U}} \\ &= \mathbf{P}\mathbf{F}^{\mathrm{T}} : \boldsymbol{\Omega}_{\mathbf{R}} + \mathbf{R}^{\mathrm{T}}\mathbf{P} : \dot{\mathbf{U}} \\ &= \boldsymbol{\tau} : \boldsymbol{\Omega}_{\mathbf{R}} + \mathbf{R}^{\mathrm{T}}\mathbf{P} : \dot{\mathbf{U}} \\ &= \mathbf{T}_{\mathrm{B}} : \dot{\mathbf{U}} \end{aligned} \tag{3.8.31}$$

so that the Biot stress is power conjugate to the right stretch tensor. Since $\mathbf{U}$ is symmetric, $\mathbf{P} : \dot{\mathbf{F}} = \operatorname{sym}\mathbf{T}_{\mathrm{B}} : \dot{\mathbf{U}}$. Also, the Biot stress is conjugate to the Biot strain tensor $\overline{\mathbf{B}} = \mathbf{U} - \mathbf{I}$ introduced in §2.2.5.

From 3.5.14 and 1.10.3h,

$$\boldsymbol{\sigma} : \mathbf{d} = \mathbf{R}\hat{\boldsymbol{\sigma}}\mathbf{R}^{\mathrm{T}} : \mathbf{d} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{d}} \tag{3.8.32}$$

so that the corotational stress is power conjugate to the **rotated deformation rate**, defined by

$$\hat{\mathbf{d}} = \mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R} \tag{3.8.33}$$

Pull Back and Push Forward

From 2.12.12-13, the double contraction of two tensors can be expressed as push-forwards and pull-backs of those tensors. For example, the stress power (per unit reference volume) in the material description is $\mathbf{S} : \dot{\mathbf{E}}$. Then, using 3.5.13, 2.12.9a and 2.5.18b, $\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$,

$$\mathbf{S} : \dot{\mathbf{E}} = \chi_*(\mathbf{S})^{\#} : \chi_*(\dot{\mathbf{E}})^{b} = \boldsymbol{\tau} : \mathbf{d} = J\boldsymbol{\sigma} : \mathbf{d} \tag{3.8.34}$$

This means that the material and spatial descriptions of the internal power can be transformed into each other using push-forward and pull-back operations.

Similarly, pulling back the corotational stress and rotated deformation rate to the intermediate configuration of Fig. 2.10.8, using 2.12.13, 2.12.27,

$$\boldsymbol{\sigma} : \mathbf{d} = \chi_*^{-1}(\boldsymbol{\sigma})^{\#}{}_{\mathbf{R}(\mathbf{g})} : \chi_*^{-1}(\mathbf{d})^{b}{}_{\mathbf{R}(\mathbf{g})} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{d}} \tag{3.8.35}$$

The stress power in terms of spatial tensors can also be expressed as a derivative of a tensor, using the Lie derivative. From 2.12.42, the Lie derivative of the Euler-Almansi strain is the rate of deformation and hence (note that there is no universal function whose derivative is **d**), so

$$J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \mathrm{L}_{\mathrm{v}}^{b}\mathbf{e} \tag{3.8.36}$$

3.8.7 Problems

1. Show that the rate of internal energy per unit reference volume $J\boldsymbol{\sigma} : \mathbf{d}$ is equivalent to $\mathbf{S} : \dot{\mathbf{E}}$ (without using push-forwards/pull-backs).

3.9 The Principles of Virtual Work and Power

The principle of virtual work was introduced and discussed in Part I, §5.5. As mentioned there, it is yet another re-statement of the work – energy principle, only it is couched in terms of virtual displacements, and the principle of virtual power to be introduced below is an equivalent statement based on **virtual velocities**.

On the one hand, the principle of virtual work/power can be regarded as the fundamental law of dynamics for a continuum, and from it can be derived the equations of motion. On the other hand, one can regard the principle of linear momentum as the fundamental law, derive the equations of motion, and hence derive the principle of virtual work.

3.9.1 Overview of The Principle of Virtual Work

Consider a material under the action of external forces: body forces **b** and tractions **t**. The body undergoes a displacement $\mathbf{u}(\mathbf{x})$ due to these forces and now occupies its current configuration, Fig. 3.9.1. The problem is to find this displacement function $\mathbf{u}$.

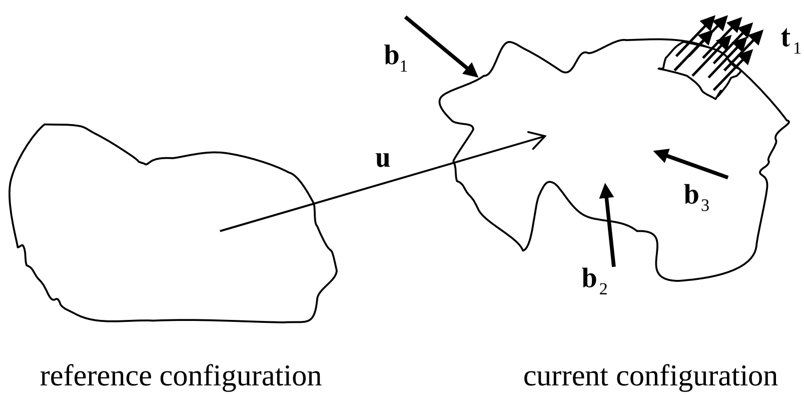

Figure 3.9.1: a material displacing to its current configuration under the action of body forces and surface forces

Imagine the material to undergo a small displacement $\delta\mathbf{u}$ from the *current configuration*, Fig. 3.9.2, $\delta\mathbf{u}$ not necessarily constant throughout the body; $\delta\mathbf{u}$ is a virtual displacement, meaning that it is an *imaginary* displacement, and in no way is it related to the applied external forces – *it does not actually occur physically*.

As each material particle moves through these virtual displacements, the external forces do virtual work δW. If the force $\mathbf{b}$ acts at position $\mathbf{x}$ and this point undergoes a virtual displacement $\delta u(\mathbf{x})$, the virtual work is $\delta W = \mathbf{b}\cdot\delta\mathbf{u}\,\Delta v$. Similarly for the surface tractions, and the total external virtual work is

$$\boxed{\delta W_{\text{ext}} = \int_v \mathbf{b}\cdot\delta\mathbf{u}\, dv + \int_s \mathbf{t}\cdot\delta\mathbf{u}\, ds = 0} \qquad \textbf{External Virtual Work} \qquad (3.9.1)$$

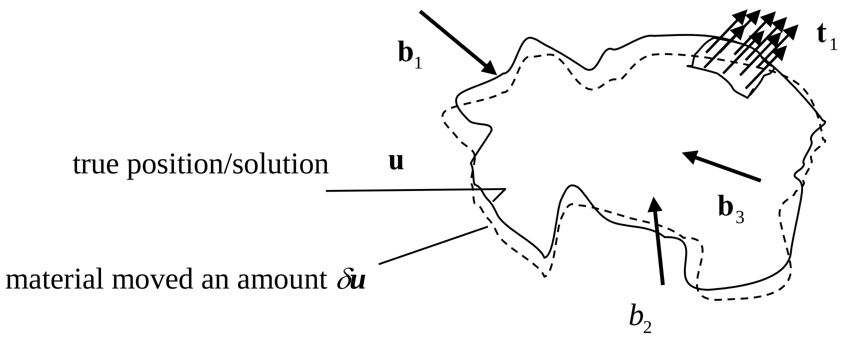

Figure 3.9.2: a virtual displacement field applied to a material in the current configuration

There is also an internal virtual work W_{int} due to the internal forces as they move through virtual displacements and a virtual kinetic energy δK. The principle of virtual work then says that

$$\delta W_{\mathrm{ext}} + \delta W_{\mathrm{int}} = \delta K \tag{3.9.2}$$

And this equation is then solved for the actual displacement **u**. Expressions for the internal virtual work and virtual kinetic energy will be derived below.

3.9.2 Derivation of The Principle of Virtual Work

As mentioned above, one can simply write down the principle of virtual work, regarding it as the fundamental principle of mechanics, and then from it derive the equations of motion. This will be done further below. To begin, though, the starting point will be the equations of motion, and from it will be derived the principle of virtual work.

Kinematically and Statically Admissible Fields

A **kinematically admissible displacement field** is defined to be one which satisfies the displacement boundary condition 3.7.7c, $\mathbf{u} = \bar{\mathbf{u}}$ on s_u (see Part I, §5.5.1). Such a displacement field would induce some stress field within the body, but this resulting stress field might not satisfy the equations of motion 3.7.7a. In other words, it might not be the actual displacement field, but it does not violate the boundary conditions.

A **statically admissible stress field** is one which satisfies the equations of motion 3.7.7a and the traction boundary conditions 3.7.7b, $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}$, on s_σ. Again, it might not be the actual stress field, since it is not specified how this stress field should be related to the actual displacement field.

Derivation from the Equations of Motion (Spatial Form)

Let $\boldsymbol{\sigma}$ be a statically admissible stress field corresponding to a kinematically admissible displacement field $\mathbf{u}$, so $\boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}$ on s_σ, $\mathbf{u} = \bar{\mathbf{u}}$ on s_u and $\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}$. Multiplying the equations of motion by $\mathbf{u}$ and integrating leads to

$$\int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}\, dv = \int_v \left(\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}\right) \cdot \mathbf{u}\, dv \tag{3.9.3}$$

Using the identity 1.14.16b, $\operatorname{div}(\mathbf{A}\mathbf{v}) = \mathbf{v} \cdot \operatorname{div}\mathbf{A}^{\mathrm{T}} + \operatorname{tr}(\mathbf{A}\operatorname{grad}\mathbf{v})$, 1.10.10e, $\operatorname{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathbf{A} : \mathbf{B}$, and the symmetry of stress,

$$\int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}\, dv = \int_v \left\{\operatorname{div}(\boldsymbol{\sigma}\mathbf{u}) - \boldsymbol{\sigma} : \operatorname{grad}(\mathbf{u}) + \mathbf{b} \cdot \mathbf{u}\right\} dv \tag{3.9.4}$$

and the divergence theorem 1.14.22c and Cauchy's law lead to

$$\int_s \mathbf{t} \cdot \mathbf{u}\, ds + \int_v \mathbf{b} \cdot \mathbf{u}\, dv = \int_v \boldsymbol{\sigma} : \operatorname{grad}\mathbf{u}\, dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}\, dv \tag{3.9.5}$$

Splitting the surface integral into one over $s_{\mathbf{u}}$ and one over $s_{\boldsymbol{\sigma}}$ gives

$$\int_{s_{\mathbf{u}}} \mathbf{t} \cdot \bar{\mathbf{u}}\, ds + \int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \mathbf{u}\, ds + \int_v \mathbf{b} \cdot \mathbf{u}\, dv = \int_v \boldsymbol{\sigma} : \operatorname{grad}\mathbf{u}\, dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}\, dv \tag{3.9.6}$$

Next, consider a second kinematically admissible displacement field $\mathbf{u}^*$, so $\mathbf{u}^* = \bar{\mathbf{u}}$ on s_u, which is completely arbitrary, in the sense that it is unrelated to either $\boldsymbol{\sigma}$ or $\mathbf{u}$. This time multiplying $\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}$ across by $\mathbf{u}^*$, and following the same procedure, one arrives at

$$\int_{s_{\mathbf{u}}} \mathbf{t} \cdot \bar{\mathbf{u}}\, ds + \int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \mathbf{u}^* ds + \int_v \mathbf{b} \cdot \mathbf{u}^* dv = \int_v \boldsymbol{\sigma} : \operatorname{grad}\mathbf{u}^* dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}^* dv \tag{3.9.7}$$

Let $\delta\mathbf{u} = \mathbf{u}^* - \mathbf{u}$, so the difference between $\mathbf{u}^*$ and $\mathbf{u}$ is infinitesimal, then subtracting 3.9.6 from 3.9.7 gives the principle of virtual work,

$$\boxed{\int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \delta\mathbf{u}\, ds + \int_v \mathbf{b} \cdot \delta\mathbf{u}\, dv = \int_v \boldsymbol{\sigma} : \operatorname{grad}(\delta\mathbf{u})\, dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dv}$$

Principle of Virtual Work (spatial form) (3.9.8)

Note that since $\mathbf{u}, \mathbf{u}^*$, are kinematically admissible, $\delta\mathbf{u} = \mathbf{u}^* - \mathbf{u} = \mathbf{o}$ on s_u.

If one considers the $\mathbf{u}$ in 3.9.8 to be the actual displacement of the body, then $\delta\mathbf{u}$ can be considered to be a virtual displacement from the current configuration, Fig. 3.9.2. Again, it is emphasized that this virtual displacement leaves the stress, body force and applied traction unchanged.

One also has the transformed initial conditions: from 3.7.7d-e,

$$\int_v \mathbf{u}(\mathbf{x},\mathbf{t})_{t=0} \cdot \delta\mathbf{u}dv = \int_v \mathbf{u}_0(\mathbf{x}) \cdot \delta\mathbf{u}dv$$
$$\int_v \dot{\mathbf{u}}(\mathbf{x},\mathbf{t})_{t=0} \cdot \delta\mathbf{u}dv = \int_v \dot{\mathbf{u}}_0(\mathbf{x}) \cdot \delta\mathbf{u}dv \qquad (3.9.9)$$

Eqns. 3.9.8 and 3.9.9 together constitute the **weak form** of the initial BVP 3.7.7.

The principle of virtual work can be grouped into three separate terms: the external virtual work:

$$\boxed{\delta W_{\text{ext}} = \int_{s_\sigma} \bar{\mathbf{t}} \cdot \delta\mathbf{u}ds + \int_v \mathbf{b} \cdot \delta\mathbf{u}dv} \quad \textbf{External Virtual Work} \qquad (3.9.10)$$

the internal virtual work,

$$\boxed{\delta W_{\text{int}} = -\int_v \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{u})dv} \quad \textbf{Internal Virtual Work} \qquad (3.9.11)$$

and the virtual kinetic energy,

$$\boxed{\delta K = \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}dv} \quad \textbf{Virtual Kinetic Energy} \qquad (3.9.12)$$

corresponding to the statement 3.9.2.

Derivation from the Equations of Motion (Material Form)

The derivation in the spatial form follows exactly the same lines as for the spatial form.

This time, let $\mathbf{P}$ be a statically admissible stress field corresponding to a kinematically admissible displacement field $\mathbf{U}$, so $\mathbf{PN} = \overline{\mathbf{T}}$ on $S_\mathbf{P}$, $\mathbf{U} = \overline{\mathbf{U}}$ on $S_\mathbf{u}$ and $\text{div}\mathbf{P} + \mathbf{B} = \rho_0\ddot{\mathbf{U}}$. This time one arrives at

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\mathbf{U}dS + \int_V \mathbf{B} \cdot \delta\mathbf{U}dV = \int_V \mathbf{P} : \text{Grad}(\delta\mathbf{U})dV + \int_V \rho_0\ddot{\mathbf{U}} \cdot \delta\mathbf{U}dV \qquad (3.9.13)$$

Again, one can consider $\mathbf{U}$ to be the actual displacement of the body, so that $\delta\mathbf{U}$ represents a virtual displacement from the current configuration. With

$$\mathbf{U} = \mathbf{x} - \mathbf{X}$$
$$\delta\mathbf{U} = \delta\mathbf{x} - \delta\mathbf{X} = \delta\mathbf{x} \qquad (3.9.14)$$

the virtual work equation can be expressed in terms of the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$,

$$\boxed{\int_{S_{\mathbf{P}}} \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi} dS + \int_V \mathbf{B} \cdot \delta\boldsymbol{\chi} dV = \int_V \mathbf{P} : \mathrm{Grad}(\delta\boldsymbol{\chi}) dV + \int_V \rho_0 \ddot{\mathbf{U}} \cdot \delta\boldsymbol{\chi} dV}$$

Principle of Virtual Work (material form) (3.9.15)

3.9.3 Principle of Virtual Work in terms of Strain Tensors

The principle of virtual work, in particular the internal virtual work term, can be expressed in terms of strain tensors.

Spatial Form

Using the commutative property of the variation 2.13.2, the term $\mathrm{grad}(\delta\mathbf{u})$ in the internal virtual work expression 3.9.8 can be written as

$$\begin{aligned} \mathrm{grad}(\delta\mathbf{u}) &= \frac{1}{2}\left(\mathrm{grad}(\delta\mathbf{u}) + \left(\mathrm{grad}(\delta\mathbf{u})\right)^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathrm{grad}(\delta\mathbf{u}) - \left(\mathrm{grad}(\delta\mathbf{u})\right)^{\mathrm{T}}\right) \\ &= \delta\frac{1}{2}\left(\mathrm{grad}\mathbf{u} + \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\right) + \delta\frac{1}{2}\left(\mathrm{grad}\mathbf{u} - \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\right) \\ &= \delta\boldsymbol{\varepsilon} + \delta\boldsymbol{\Omega} \end{aligned} \qquad (3.9.16)$$

where $\boldsymbol{\varepsilon}$ is the (symmetric) small strain tensor and $\boldsymbol{\Omega}$ is the (skew-symmetric) small rotation tensor, Eqn 2.7.2. Using the fact that the double contraction of a symmetric tensor ($\boldsymbol{\sigma}$) and a skew-symmetric one ($\boldsymbol{\Omega}$) is zero, 1.10.31c, one has

$$\delta W_{\mathrm{int}} = -\int_v \boldsymbol{\sigma} : \mathrm{grad}(\delta\mathbf{u}) dv = -\int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dv \qquad (3.9.17)$$

Thus the stresses do internal virtual work along the virtual strains $\delta\boldsymbol{\varepsilon}$. One has

$$\int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \delta\mathbf{u} ds + \int_v \mathbf{b} \cdot \delta\mathbf{u} dv = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dv + \int_v \rho \ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv \qquad (3.9.18)$$

Note that, although the small strain has been introduced here, this formulation is not restricted to small-strain theory. It is only the virtual strains that must be infinitesimal – there is no restriction on the magnitude of the *actual* strains.

From 2.13.15, the Lie-variation of the Euler-Almansi strain **e** is $\delta_{\mathrm{L}}\mathbf{e} = \delta\boldsymbol{\varepsilon}$, so the internal; virtual work can be expressed as

$$\delta W_{\mathrm{int}} = -\int_v \boldsymbol{\sigma} : \delta_{\mathrm{L}}\mathbf{e}\, dv \qquad (3.9.19)$$

Material Form

From Eqn. 3.9.15 and Eqn. 2.13.9,

$$\delta W_{\text{int}} = -\int_V \mathbf{P} : \delta\mathbf{F}\, dV \tag{3.9.20}$$

so

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi}\, dS + \int_V \mathbf{B} \cdot \delta\boldsymbol{\chi}\, dV = \int_V \mathbf{P} : \delta\mathbf{F}\, dV + \int_V \rho_0 \ddot{\mathbf{U}} \cdot \delta\boldsymbol{\chi}\, dV \tag{3.9.21}$$

Derivation of the Material Form directly from the Spatial Form

To transform the spatial form of the virtual work equation into the material form, first note that, with 3.9.14b,

$$\boldsymbol{\sigma} : \text{grad}(\delta\mathbf{u}) = \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{x}) \tag{3.9.22}$$

Then, using 2.2.8b, $\text{grad}\mathbf{v} = \text{Grad}\mathbf{V}\,\mathbf{F}^{-1}$, 2.13.9, $\delta\mathbf{F} = \text{Grad}(\delta\mathbf{u})$, 1.10.3h, $\mathbf{A} : (\mathbf{BC}) = (\mathbf{AC}^{\mathrm{T}}) : \mathbf{B}$, and 3.5.10, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$,

$$\begin{aligned} \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{x}) &= \boldsymbol{\sigma} : \left(\text{Grad}(\delta\mathbf{x})\mathbf{F}^{-1}\right) \\ &= \boldsymbol{\sigma} : \left(\delta\mathbf{F}\mathbf{F}^{-1}\right) \\ &= \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right) : \delta\mathbf{F} \\ &= \left(J^{-1}\mathbf{P}\right) : \delta\mathbf{F} \end{aligned} \tag{3.9.23}$$

which converts 3.9.17 into 3.9.120.

Also, again comparing 3.9.17 and 3.9.20, using the trace properties 1.10.10, and Eqns. 3.5.9 and 2.13.11b,

$$\mathbf{P} : \delta\mathbf{F} = J\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = J\text{tr}\left(\boldsymbol{\sigma}\delta\boldsymbol{\varepsilon}\right) = J\text{tr}\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\delta\boldsymbol{\varepsilon}\mathbf{F}\right) = J\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right) : \left(\mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\right) = \mathbf{S} : \delta\mathbf{E} \tag{3.9.24}$$

and so the internal work can also be expressed as an integral of $\mathbf{S} : \delta\mathbf{E}$ over the reference volume.

The Internal Virtual Work and Work Conjugate Tensors

The expressions for stress power 3.8.15, 3.8.29, and internal virtual work are very similar. For the material description, the time derivatives in the former are simply replaced with the variation to get the latter:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} \quad \rightarrow \quad \mathbf{P} : \delta\mathbf{F} = \mathbf{S} : \delta\mathbf{E} \tag{3.9.25}$$

For spatial tensors, the rate of strain tensor, e.g. **d**, is replaced with a Lie variation 2.13.14. For example, $J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \mathrm{L}_{\mathrm{v}}^{b}\mathbf{e}$ (see 2.12.41-42) becomes:

$$J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \mathrm{L}_{\mathrm{v}}^{b}\mathbf{e} \quad \rightarrow \quad J\boldsymbol{\sigma} : \delta_{\mathrm{L}}\mathbf{e} \tag{3.9.26}$$

3.9.4 Derivation of the Strong Form from the Weak Form

Just as the strong form (equations of motion and boundary conditions) was converted into the weak form (principle of virtual work), the weak form can be converted back into the strong form. For example,

$$\begin{aligned}\int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv &= \int_v \boldsymbol{\sigma} : \delta \operatorname{grad}\mathbf{u} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv \\ &= \int_v \left\{ \operatorname{div}(\boldsymbol{\sigma} \cdot \delta\mathbf{u}) - (\operatorname{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} \right\} dv \\ &= \int_s \mathbf{t} \cdot \delta\mathbf{u} ds - \int_v \left\{ (\operatorname{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} \right\} dv \\ &= \int_{s_\sigma} \mathbf{t} \cdot \delta\mathbf{u} ds - \int_v \left\{ (\operatorname{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} \right\} dv \end{aligned} \tag{3.9.27}$$

and the last line follows from the fact that $\delta\mathbf{u} = \mathbf{o}$ on $s_\mathbf{u}$. Thus the weak form now reads

$$\int_{s_\sigma} (\mathbf{t} - \bar{\mathbf{t}}) \cdot \delta\mathbf{u} ds - \int_v (\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} dv = 0 \tag{3.9.28}$$

and, since $\delta\mathbf{u}$ is arbitrary, one finds that the expressions in the parentheses are zero, and so 3.7.7 is recovered.

3.9.5 Conservative Systems

Thus far, no assumption has been made about the nature of the internal forces acting in the material. Indeed, the principle of virtual work applies to all types of materials.

Now, however, attention is restricted to the special case where the system is conservative, in the sense that the work done by the external loads and the internal forces can be written in terms of potential energy functions[1]. Further, for brevity, assume also that the material is in static equilibrium, i.e. the kinetic energy term is zero.

In other words, it is assumed that the internal virtual work term can be expressed in the form of a virtual potential energy function:

$$\int_v \boldsymbol{\sigma} : \operatorname{grad}(\delta\mathbf{u}) dv = \int_v \delta U dv \tag{3.9.29}$$

Here, U is considered to be a function of $\mathbf{u}$, and the variation is to be understood as in Eqn. 2.13.5, $\delta U(\mathbf{u}, \delta\mathbf{u}) \equiv \partial_\mathbf{u} U[\delta\mathbf{u}]$.

If the loads can be regarded as functions of **u** only then, since they are conservative, they may be written as the gradient of a scalar potential:

[1] The external loads being conservative would exclude, for example, cases of frictional loading

$$\mathbf{b} = -\frac{\partial U_b}{\partial \mathbf{u}}, \quad \bar{\mathbf{t}} = -\frac{\partial U_t}{\partial \mathbf{u}} \tag{3.9.30}$$

Then, with

$$\delta U_b = \frac{\partial U_b}{\partial \mathbf{u}} \cdot \delta \mathbf{u}, \quad \delta U_t = \frac{\partial U_t}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \tag{3.9.31}$$

and using the commutative property 2.13.3 of the variational operator, one arrives at

$$\delta \left\{ \int_v U dv + \int_{s_\sigma} U_t ds + \int_v U_b dv \right\} = \delta \overline{U}(\mathbf{u}) = 0 \tag{3.9.32}$$

The quantity inside the brackets is the total potential energy of the system. This statement is the **principle of stationary potential energy**: the value of the quantity inside the parentheses, i.e. $\overline{U}(\mathbf{u})$, is stationary at the true solution $\mathbf{u}$.

Eqn. 3.9.32 is an example of a **Variational Principle**, that is, a principle expressed in the form of a variation of a functional. Note that the principle of virtual work in the form 3.9.8 is not a variational principle, since it is not expressed as the variation of one functional.

Body Forces

Body forces can usually be expressed in the form 3.9.30. For example, with gravity loading, $\mathbf{b} = \rho \mathbf{g}$, where $\mathbf{g}$ is the constant acceleration due to gravity. Then $U_b = -\rho \mathbf{g} \cdot \mathbf{u}$ ($\delta \mathbf{b} = \delta \mathbf{g} = \mathbf{o}$ and $\mathbf{b} \cdot \delta \mathbf{u} = \delta(\mathbf{b} \cdot \mathbf{u})$, so $\int \mathbf{b} \cdot \delta \mathbf{u} dv = \delta \int \mathbf{b} \cdot \mathbf{u} dv$).

Material Form

In the material form, one again has a stationary principle if one can write $\mathbf{B} = -\partial U_B(\mathbf{U}) / \partial \mathbf{U}$, $\overline{\mathbf{T}} = -\partial U_T(\mathbf{U}) / \partial \mathbf{U}$ (or, equivalently, replacing $\mathbf{U}$ with the motion $\boldsymbol{\chi}$). In the case of dead loading, §3.7.1, $\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X})$ is independent of the motion so (similar to the case of gravity loading above) $U_T = -\overline{\mathbf{T}} \cdot \mathbf{u}$ with $\delta \overline{\mathbf{T}} = \mathbf{o}$ and $\int \overline{\mathbf{T}} \cdot \delta \boldsymbol{\chi} dV = \delta \int \overline{\mathbf{T}} \cdot \boldsymbol{\chi} dV$.

Deformation Dependent Traction

In many practical cases, the traction will depend on not only the motion, but also the strain. In that case, one can write

$$\int_{s_\sigma} \bar{\mathbf{t}} \cdot \delta \mathbf{u} ds = \int_s \overline{\boldsymbol{\sigma}} \mathbf{n} \cdot \delta \mathbf{u} ds = \int_s \mathbf{n} \overline{\boldsymbol{\sigma}} \delta \mathbf{u} ds = \int_v \operatorname{div}(\overline{\boldsymbol{\sigma}} \delta \mathbf{u}) dv = \int_v (\operatorname{div} \overline{\boldsymbol{\sigma}} \cdot \delta \mathbf{u} + \overline{\boldsymbol{\sigma}} : \operatorname{grad} \delta \mathbf{u}) dv$$

One might be able to then introduce a scalar function ϕ such that

$$\delta\phi(\mathbf{u},\boldsymbol{\varepsilon}) = \frac{\partial\phi}{\partial\mathbf{u}}\cdot\delta\mathbf{u} + \frac{\partial\phi}{\partial\boldsymbol{\varepsilon}} : \delta\boldsymbol{\varepsilon} \quad \text{with} \quad \frac{\partial\phi}{\partial\mathbf{u}} = \mathrm{div}\overline{\boldsymbol{\sigma}}, \;\; \frac{\partial\phi}{\partial\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\sigma}} \tag{3.9.33}$$

In the material form, one would have $\overline{\mathbf{T}} = \overline{\mathbf{P}}\mathbf{N}$ with

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}}\cdot\delta\boldsymbol{\chi}dS = \int_V \left(\mathrm{Div}\,\overline{\mathbf{P}}\cdot\delta\boldsymbol{\chi} + \overline{\mathbf{P}} : \mathbf{F}\right)dV$$

and then one might be able to introduce a scalar function $\phi(\boldsymbol{\chi},\mathbf{F})$ such that

$$\delta\phi = \frac{\partial\phi}{\partial\boldsymbol{\chi}}\cdot\delta\boldsymbol{\chi} + \frac{\partial\phi}{\partial\mathbf{F}} : \delta\mathbf{F} \quad \text{with} \quad \frac{\partial\phi}{\partial\boldsymbol{\chi}} = \mathrm{Div}\overline{\mathbf{P}}, \;\; \frac{\partial\phi}{\partial\mathbf{F}} = \overline{\mathbf{P}} \tag{3.9.34}$$

For example, considering again the fluid pressure example of §3.7.1, one can let $\phi = -pJ$ so that, using 1.15.7, $\partial\phi/\partial\mathbf{F} = -pJ\mathbf{F}^{-\mathrm{T}}$. Then $\overline{\mathbf{P}} = -p\mathbf{F}^{-\mathrm{T}} = \partial\phi/\partial\mathbf{F}\big|_{J=1}$, $\mathrm{Div}\overline{\mathbf{P}} = \rho g\mathbf{E}_2$ and $\int \overline{\mathbf{T}}\cdot\delta\boldsymbol{\chi}dS = -\delta\int pJdV$.

3.9.6 The Principle of Virtual Power

The principle of virtual power is similar to the principle of virtual work, the only difference between them being that a virtual velocity $\delta\mathbf{v}$ is used in the former rather than a virtual displacement. To derive the virtual power equation, multiply the equations of motion by the virtual velocity function, and integrate over the current configuration, giving

$$\begin{aligned}\int_v \rho\frac{d\mathbf{v}}{dt}\cdot\delta\mathbf{v}dv = \int_v (\mathrm{div}\boldsymbol{\sigma} + \mathbf{b})\cdot\delta\mathbf{v}dv &= \int_v \left\{\mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{v}) - \boldsymbol{\sigma} : \frac{\partial(\delta\mathbf{v})}{\partial\mathbf{x}} + \mathbf{b}\cdot\delta\mathbf{v}\right\}dv \\ &= \int_v \left\{\mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{v}) - \boldsymbol{\sigma} : \delta\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + \mathbf{b}\cdot\delta\mathbf{v}\right\}dv \\ &= \int_s \mathbf{t}\cdot\delta\mathbf{v}ds - \int_v \boldsymbol{\sigma} : \delta\mathbf{d}dv + \int_v \mathbf{b}\cdot\delta\mathbf{v}dv\end{aligned} \tag{3.9.35}$$

These equations are identical to the mechanical balance equations 3.8.16, except that the actual velocity is replaced with a virtual velocity. The term $-\int_v \boldsymbol{\sigma} : \delta\mathbf{d}dv$ is called the **internal virtual power**.

Note that here, unlike the virtual displacement function in the work equation, the virtual velocity does not have to be infinitesimal. This can be seen more clearly if one derives this equation directly from the virtual work equation. If the infinitesimal virtual displacement $\delta\mathbf{u}$ occurs over an infinitesimal time interval δt, the virtual velocity is the finite quantity $\delta\mathbf{u}/\delta t$, which here is labelled $\delta\mathbf{v}$. The virtual power equation can thus be obtained by dividing the virtual work equation through by δt.

Again, supposing that the velocities are specified over that part of the surface s_v and tractions over $s_{\boldsymbol{\sigma}}$, the principle of virtual power can be written for the case of a kinematically admissible virtual velocity field:

$$\boxed{\int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \delta \mathbf{v} ds + \int_v \mathbf{b} \cdot \delta \mathbf{v} dv = \int_v \boldsymbol{\sigma} : \delta \mathbf{d} dv + \int_v \rho \frac{d\mathbf{v}}{dt} \cdot \delta \mathbf{v} dv} \quad \textbf{Principle of Virtual Power} \quad (3.9.36)$$

In words, the principle of virtual power states that *at any time t, the total virtual power of the external, internal and inertia forces is zero in any admissible virtual state of motion*.

3.9.7 Linearisation of the Internal Virtual Work

In order to solve the virtual work equations in anything but the most simple cases, one must apply some approximate numerical method. This will usually involve linearising the non-linear virtual work equations. To this end, the internal virtual work term will be linearised in what follows.

Material Description

In the material description, one has

$$\delta W_{\text{int}} = \int_V \mathbf{S}(\mathbf{E}(\mathbf{u})) : \delta \mathbf{E}(\mathbf{u}) dV \qquad (3.9.37)$$

in which the Green-Lagrange strain is considered to be a function of the displacement, Eqn. 2.2.46, and the PK2 stress is a function of the Green-Lagrange strain; the precise functional dependence of **S** on **E** will depend on the material under study (see Part IV).

The linearisation of the variation of a function is given by (see §2.13.2)

$$\mathrm{L}\ \delta W_{\text{int}}(\mathbf{u}, \Delta \mathbf{u}) = \delta W_{\text{int}}(\mathbf{u}) + \Delta \delta W_{\text{int}}(\mathbf{u}, \Delta \mathbf{u}) \qquad (3.9.38)$$

where

$$\begin{aligned}
\Delta \delta W_{\text{int}}(\mathbf{u}, \Delta \mathbf{u}) &= \partial_{\mathbf{u}} \delta W_{\text{int}}[\Delta \mathbf{u}] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta W_{\text{int}}(\mathbf{u} + \varepsilon \Delta \mathbf{u}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_V \mathbf{S}(\mathbf{E}(\mathbf{u} + \varepsilon \Delta \mathbf{u})) : \delta \mathbf{E}(\mathbf{u} + \varepsilon \Delta \mathbf{u}) dV \\
&= \int_V \left\{ \mathbf{S}(\mathbf{E}(\mathbf{u})) : \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta \mathbf{E}(\mathbf{u} + \varepsilon \Delta \mathbf{u}) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{S}(\mathbf{E}(\mathbf{u} + \varepsilon \Delta \mathbf{u})) : \delta \mathbf{E}(\mathbf{u}) \right\} dV \\
&= \int_V \{ \mathbf{S}(\mathbf{E}(\mathbf{u})) : \Delta \delta \mathbf{E}(\mathbf{u}, \Delta \mathbf{u}) + \Delta \mathbf{S}(\mathbf{E}(\mathbf{u}, \Delta \mathbf{u})) : \delta \mathbf{E}(\mathbf{u}) \} dV
\end{aligned}$$

(3.9.39)

The linearization of the variation of the Green-Lagrange strain is given by 2.13.24, $\Delta\delta\mathbf{E} = \mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)$. With the PK2 stress symmetric, one has, with 1.10.3h, 1.10.31c,

$$\begin{aligned}\mathbf{S}(\mathbf{E}(\mathbf{u})):\Delta\delta\mathbf{E}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{S}(\mathbf{E}(\mathbf{u})):\mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)\\ &= \mathbf{S}(\mathbf{E}(\mathbf{u})):(\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\\ &= \mathrm{Grad}\delta\mathbf{u}:(\mathrm{Grad}\Delta\mathbf{u})\mathbf{S}(\mathbf{E}(\mathbf{u}))\end{aligned} \tag{3.9.40}$$

For the second term in 3.9.39, from 2.13.22, the variation of **E** is

$$\begin{aligned}\delta\mathbf{E} &= \tfrac{1}{2}\left[(\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right]\\ &= \tfrac{1}{2}\left[\left(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right]\\ &= \mathrm{sym}\left(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)\end{aligned} \tag{3.9.41}$$

What remains is the calculation of the linearisation of the PK2 stress. One has using the chain rule,

$$\begin{aligned}\Delta\mathbf{S}(\mathbf{E}(\mathbf{u},\Delta\mathbf{u})) &= \partial_{\mathbf{u}}\mathbf{S}[\Delta\mathbf{u}]\\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{S}(\mathbf{E}(\mathbf{u}+\varepsilon\Delta\mathbf{u}))\\ &= \frac{\partial\mathbf{S}(\mathbf{E})}{\partial\mathbf{E}}:\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{E}((\mathbf{u}+\varepsilon\Delta\mathbf{u}))\\ &= \frac{\partial\mathbf{S}(\mathbf{E})}{\partial\mathbf{E}}:\Delta\mathbf{E}(\mathbf{u},\Delta\mathbf{u})\end{aligned} \tag{3.9.42}$$

Denote the fourth order tensor $\partial\mathbf{S}(\mathbf{E})/\partial\mathbf{E}$ by $\mathbf{C}$ and assume that it has the minor symmetries1.12.10. Then (see 3.9.41), with

$$\Delta\mathbf{E} = \mathrm{sym}\left(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}\right) \tag{3.9.43}$$

the linear increments in 3.9.38 become

$$\boxed{\begin{aligned}\Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \{\mathrm{Grad}\delta\mathbf{u}:(\mathrm{Grad}\Delta\mathbf{u})\mathbf{S}(\mathbf{E}(\mathbf{u}))\\ &\qquad\qquad + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}:\mathbf{C}:\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}\}dV\\ \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \left\{\frac{\partial\delta u_i}{\partial X_b}\frac{\partial\Delta u_i}{\partial X_d}S_{bd} + F_{ka}\frac{\partial\delta u_k}{\partial X_b}C_{abcd}F_{jc}\frac{\partial\Delta u_j}{\partial X_d}\right\}dV\\ &= \int_V \frac{\partial\delta u_i}{\partial X_b}\{\delta_{ij}S_{bd} + F_{ia}F_{jc}C_{abcd}\}\frac{\partial\Delta u_j}{\partial X_d}dV\end{aligned}} \tag{3.9.44}$$

The first term is due to the current stress and is called the (**initial**) **stress contribution**. The second term depends on the material properties and is called the **material contribution**. Solution formulations based on 3.9.44 are called **total Lagrangian**.

Spatial Description

The spatial description can be obtained by pushing forward the material description. First note that the linearization of the Kirchhoff stress is, from 3.5.13,

$$\begin{aligned} \mathrm{L}\ \boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= \chi_*(\mathrm{L}\ \mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^\# \\ &= \chi_*(\mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^\# + \chi_*(\Delta\ \mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^\# \\ &= \boldsymbol{\tau}(\mathbf{u}) + \mathbf{F}\Delta\mathbf{S}(\mathbf{u},\Delta\mathbf{u})\mathbf{F}^{\mathrm{T}} \end{aligned} \tag{3.9.45}$$

so that, as in the derivation of the material term in 3.9.44, and using 2.4.8,

$$\begin{aligned} \Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{F}\left(\mathsf{C} : \mathbf{F}^{\mathrm{T}}\mathrm{grad}\Delta\mathbf{u}\mathbf{F}\right)\mathbf{F}^{\mathrm{T}} \\ \Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= F_{ia}F_{jb}F_{kc}F_{ld}C_{abcd}\frac{\partial\Delta u_k}{\partial x_l} \end{aligned} \tag{3.9.46}$$

Define the fourth-order spatial tensor c through

$$c_{ijkl} = J^{-1}F_{ia}F_{jb}F_{kc}F_{ld}C_{abcd} \tag{3.9.47}$$

so that

$$\Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) = J\,\mathsf{c} : \mathrm{grad}\Delta\mathbf{u} \tag{3.9.48}$$

Then, from 3.9.39,

$$\begin{aligned} \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \left\{\chi_*(\mathbf{S})^\# : \chi_*(\Delta\delta\mathbf{E})^b + \chi_*(\Delta\mathbf{S})^\# : \chi_*(\delta\mathbf{E})^b\right\}dV \\ &= \int_V \left\{\boldsymbol{\tau} : \mathrm{sym}\left((\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{grad}\delta\mathbf{u}\right) + J\,\mathsf{c} : \mathrm{grad}\Delta\mathbf{u} : \mathrm{sym}(\mathrm{grad}\delta\mathbf{u})\right\}dV \\ &= \int_v \left\{\boldsymbol{\sigma} : (\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{grad}\delta\mathbf{u} + \mathsf{c} : \mathrm{grad}\Delta\mathbf{u} : \mathrm{grad}\delta\mathbf{u}\right\}dv \\ &= \int_v \left\{\mathrm{grad}\delta\mathbf{u} : \mathrm{grad}\Delta\mathbf{u}\,\boldsymbol{\sigma} + \mathrm{grad}\delta\mathbf{u} : \mathsf{c} : \mathrm{grad}\Delta\mathbf{u}\right\}dv \\ \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_v \left\{\frac{\partial\delta u_a}{\partial x_b}\frac{\partial\Delta u_a}{\partial x_d}\sigma_{bd} + \frac{\partial\delta u_a}{\partial x_b}c_{abcd}\frac{\partial\Delta u_c}{\partial x_d}\right\}dv \\ &= \int_v \frac{\partial\delta u_a}{\partial x_b}\left\{\delta_{ac}\sigma_{bd} + c_{abcd}\right\}\frac{\partial\Delta u_c}{\partial x_d}dv \end{aligned} \tag{3.9.49}$$

Solution formulations based on 3.9.49 are called **updated-Lagrangian**.

3.10 Convected Coordinates

Some of the important results from sections 3.1-3.9 are now re-expressed in terms of convected coordinates. As before, any relations expressed in symbolic form hold also in the convected coordinate system.

3.10.1 The Stress Tensors

Traction and Stress Components

Consider a differential parallelepiped element in the current configuration bounded by the coordinate curves as in Fig. 3.10.1 (see Fig. 1.16.2). The bounding vectors are $d\Theta^1\mathbf{g}_1,\, d\Theta^2\mathbf{g}_2$ and $d\Theta^3\mathbf{g}_3$. The surface area $d\overline{S}_1$ of a face of the elemental parallelepiped on which Θ_1 is constant, to which $\mathbf{g}^1$ is normal, is then given by Eqn. 1.16.35,

$$d\overline{S}_1 = \sqrt{g g^{11}}\, d\Theta^2 d\Theta^3 \qquad (3.10.1)$$

and similarly for the other surfaces.

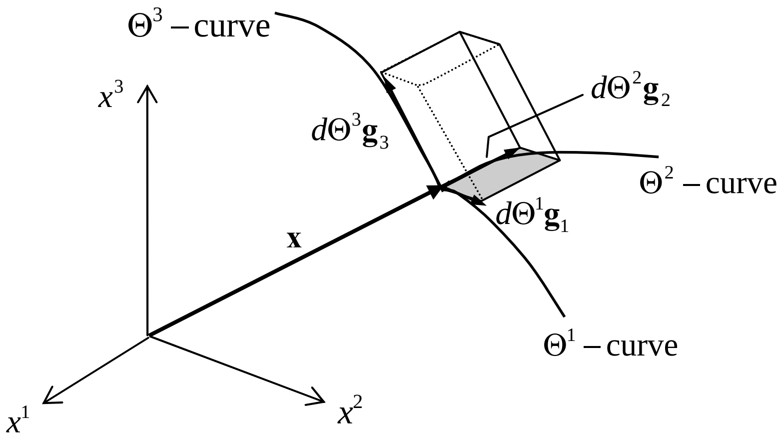

Figure 3.10.1: vector elements bounding surface elements

The **positive side** of a face is defined as that whose outward normal is in the direction of the associated contravariant base vector. The unit normal $\overline{\mathbf{n}}^i$ to a positive side is the same as the unit contravariant base vector; as in Eqn. 1.16.14,

$$\overline{\mathbf{n}}^i = \hat{\mathbf{g}}^i = \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} \qquad (\text{no sum}) \qquad (3.10.2)$$

Let the force $d\mathbf{F}^i$ acting on the surface element with normal $\overline{\mathbf{n}}^i$ be $d\overline{S}_i \mathbf{t}^{\langle i \rangle}$ (no sum over i), Fig. 3.10.2, so that $\mathbf{t}^{\langle i \rangle}$ is the traction (force per unit area).

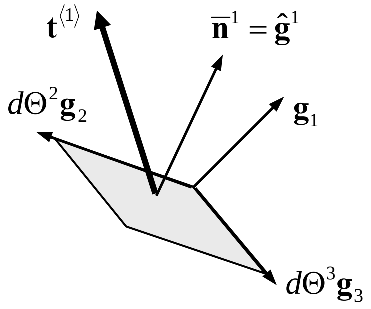

Figure 3.10.2: traction acting on a surface element

The components of $\mathbf{t}^{\langle i\rangle}$ along the unit covariant base vectors are denoted by $\sigma^{\langle ji\rangle}$:

$$\mathbf{t}^{\langle i\rangle} = \sigma^{\langle ji\rangle}\hat{\mathbf{g}}_j = \sigma^{\langle ji\rangle}\frac{1}{\sqrt{g_{jj}}}\mathbf{g}_j \tag{3.10.3}$$

with no sum over the j in the $\sqrt{g_{jj}}$ term; $\sigma^{\langle ji\rangle}$ are called the **physical stress components**, Fig. 3.10.3.

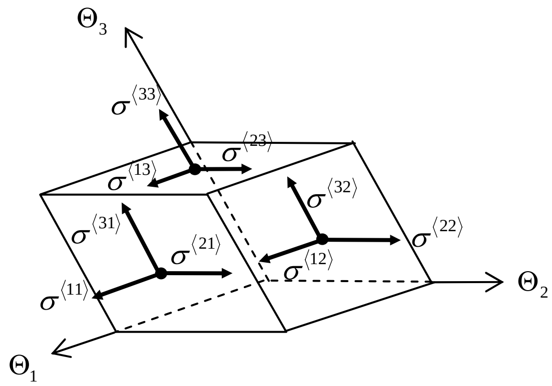

Figure 3.10.3: physical stress components

Introduce now a new vector $\mathbf{t}^i$ defined by

$$\mathbf{t}^i = \sqrt{g^{ii}}\,\mathbf{t}^{\langle i\rangle} \qquad \text{(no sum over } i\text{)} \tag{3.10.4}$$

It will be shown that this vector is contravariant, that is, transforms between coordinate systems according to 1.17.3a (and so $\mathbf{t}^{\langle i\rangle}$ does not satisfy the vector transformation rule, hence the superscript in pointed brackets). The components of $\mathbf{t}^i$ along the covariant base vectors are denoted by σ^{ji}:

$$\mathbf{t}^i = \sigma^{ji}\mathbf{g}_j \tag{3.10.5}$$

Comparing 3.10.3-5,

$$\sigma^{\langle ji \rangle} = \sqrt{\frac{g_{jj}}{g^{ii}}}\sigma^{ji} \qquad \text{(no sum)} \tag{3.10.6}$$

Cauchy's Law and the Cauchy Stress Tensor

Cauchy's law can now be derived in the same way as in §3.3, by considering a small tetrahedral free-body, Fig. 3.10.4. The physical stress components $\sigma^{\langle ij \rangle}$ shown act on the negative sides of the surfaces and so act in directions opposite that of the corresponding components on the positive sides (a consequence of Cauchy's Lemma). It is required to determine the traction **t** in terms of the physical stress components and the unit normal **n** to the base area.

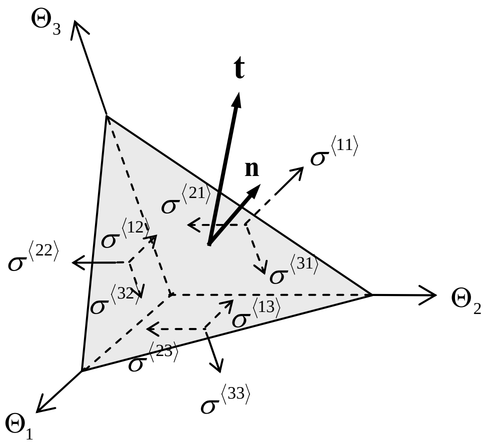

Figure 3.10.4: free body diagram of a tetrahedral portion of material

The normal to the base has components

$$\mathbf{n} = n^i \mathbf{g}_i = n_i \mathbf{g}^i \tag{3.10.7}$$

Consider the vector elements $d\mathbf{a}$ and $d\mathbf{b}$ shown in Fig. 3.10.5. Define the surface area element $d\mathbf{S}$ to be the vector with magnitude equal to twice the area of the tetrahedron base and in the direction of the normal to the base, so

$$\begin{aligned} \tfrac{1}{2} d\mathbf{S} = \tfrac{1}{2} dS\,\mathbf{n} &= \tfrac{1}{2}\left(d\mathbf{a} \times d\mathbf{b}\right) \\ &= \tfrac{1}{2}\left(\left(d\Theta^1 \mathbf{g}_1 - d\Theta^3 \mathbf{g}_3\right) \times \left(d\Theta^2 \mathbf{g}_2 - d\Theta^3 \mathbf{g}_3\right)\right) \\ &= \tfrac{1}{2}\left(d\Theta^1 d\Theta^2 \mathbf{g}_1 \times \mathbf{g}_2 + d\Theta^2 d\Theta^3 \mathbf{g}_2 \times \mathbf{g}_3 + d\Theta^3 d\Theta^1 \mathbf{g}_3 \times \mathbf{g}_1\right) \\ &= \tfrac{1}{2}\left(d\overline{\mathbf{S}}_1 + d\overline{\mathbf{S}}_2 + d\overline{\mathbf{S}}_3\right) \end{aligned} \tag{3.10.8}$$

where $d\overline{\mathbf{S}}_1$, $d\overline{\mathbf{S}}_2$, $d\overline{\mathbf{S}}_3$ are the surface element areas of the three coordinate sides of the parallelepiped of Fig. 3.10.1 (twice the area of the coordinate sides of the tetrahedron); from 3.10.2,

$$\begin{aligned} d\mathbf{S} &= d\overline{\mathbf{S}}_1 + d\overline{\mathbf{S}}_2 + d\overline{\mathbf{S}}_3 \\ dS\ \mathbf{n} &= d\overline{S}_i\ \overline{\mathbf{n}}^i \\ &= d\overline{S}_i \frac{1}{\sqrt{g^{ii}}}\mathbf{g}^i \end{aligned} \tag{3.10.9}$$

with no sum over the i in the $\sqrt{g^{ii}}$ term, or

$$dS\ n_i \sqrt{g^{ii}}\mathbf{g}^i = d\overline{S}_i \mathbf{g}^i \tag{3.10.10}$$

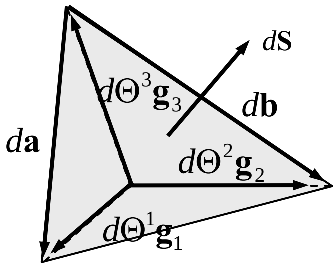

Figure 3.10.5: vector element of area for the base of the tetrahedron

The principle of linear momentum, in vector form, is then (cancelling out a factor of ½)

$$\mathbf{t}\ dS - \mathbf{t}^{\langle i \rangle} d\overline{S}_i = 0 \tag{3.10.11}$$

From 3.10.4,

$$\mathbf{t}\ dS = \mathbf{t}^i \frac{1}{\sqrt{g^{ii}}} d\overline{S}_i = \mathbf{t}^i dS n_i \tag{3.10.12}$$

and so

$$\mathbf{t} = \mathbf{t}^i n_i = \sigma^{ji} n_i \mathbf{g}_j \tag{3.10.13}$$

Defining the (symmetric) Cauchy stress tensor $\boldsymbol{\sigma}$ through

$$\boxed{\boldsymbol{\sigma} = \sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j} \qquad \textbf{Cauchy Stress Tensor} \tag{3.10.14}$$

one arrives at Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$.

The Cauchy stress is naturally a contravariant tensor because the normal vector upon which it operates to produce the traction is naturally represented in the form of a covariant vector (see 3.10.2).

Note that the stress can also be expressed in the form

$$\boldsymbol{\sigma} = \mathbf{t}^i \otimes \mathbf{g}_j \tag{3.10.15}$$

Other Stress Tensors

The PK1, PK2 and Kirchhoff stress tensors are

$$\begin{aligned} \mathbf{P} &= P^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\ \mathbf{S} &= S^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\ \boldsymbol{\tau} &= \tau^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \tag{3.10.16}$$

By definition, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, and so $\tau^{ij} = J\sigma^{ij}$. By definition, $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, and so, from 2.9.8,

$$S^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = J\sigma^{ij}\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{-1}\mathbf{g}_j = J\sigma^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \tau^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \tag{3.10.17}$$

Thus, as seen already, the Kirchhoff stress is the push-forward of the PK2 stress.

Similarly, by definition $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ and so

$$\begin{aligned} P^{ij}\mathbf{G}_i \otimes \mathbf{G}_j &= J\sigma^{kj}\mathbf{g}_k \otimes \mathbf{g}_j\mathbf{F}^{-1} \\ &= J\sigma^{kj}\mathbf{g}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}\mathbf{F}\mathbf{G}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}\left(F^{i}_{\cdot m}\mathbf{G}_i \otimes \mathbf{G}^m\right)\mathbf{G}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}F^{i}_{\cdot k}\mathbf{G}_i \otimes \mathbf{G}_j \end{aligned} \tag{3.10.18}$$

3.10.2 The Equations of Motion

The Equations of motion have been given in the symbolic form by 3.6.2 and 3.6.9. To express these in curvilinear coordinates, recall the definition of the divergence of a tensor, 1.18.28,

$$\operatorname{div}\boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial \Theta^k}\mathbf{g}^k = \sigma^{ij}\,|_k\,\left(\mathbf{g}_i \otimes \mathbf{g}_j\right)\mathbf{g}^k = \sigma^{ij}\,|_j\,\mathbf{g}_i \tag{3.10.19}$$

The spatial and material descriptions of the equations of motion are then

$$\boxed{\begin{aligned} \sigma^{ij}\,|_j\,\mathbf{g}_i + b^i\mathbf{g}_i &= \rho\frac{d\left(v^i\mathbf{g}_i\right)}{dt} \\ P^{ij}\,|_j\,\mathbf{G}_i + B^i\mathbf{G}_i &= \rho_0\frac{dV^i}{dt}\mathbf{G}_i \end{aligned}} \quad \textbf{Equations of Motion} \tag{3.10.20}$$