2 Kinematics

Kinematics is concerned with expressing in mathematical form the deformation and motion of materials. In what follows, a number of important quantities, mainly vectors and second-order tensors, are introduced. Each of these quantities, for example the velocity, deformation gradient or rate of deformation tensor, allows one to describe a particular aspect of a deforming material.

No consideration is given to what is *causing* the deformation and movement – the cause is the action of forces on the material, and these will be discussed in the next chapter.

The first section introduces the material and spatial coordinates and descriptions. The second and third sections discuss the strain tensors. The fourth, fifth and sixth sections deal with rates of deformation and rates of change of kinematic quantities. The theory is specialised to small strain deformations in section 7. The notion of objectivity and the related topic of rigid rotations are discussed in sections 8 and 9. The final sections, 10-13, deal with kinematics using the convected coordinate system, and include the important notions of push-forward, pull-back and the Lie time derivative.

2.1 Motion

2.1.1 The Material Body and Motion

Physical materials in the real world are modeled using an abstract mathematical entity called a **body**. This body consists of an infinite number of **material particles**[1]. Shown in Fig. 2.1.1a is a body *B* with material particle *P*. One distinguishes between this body and the space in which it resides and through which it travels. Shown in Fig. 2.1.1b is a certain **point** $\mathbf{x}$ in Euclidean point space *E*.

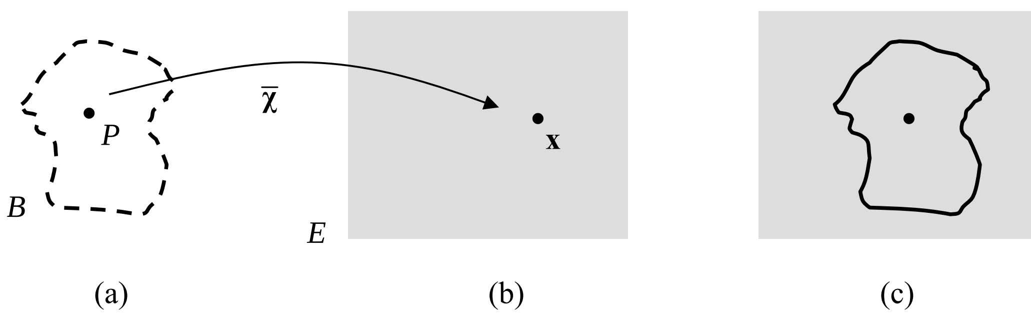

Figure 2.1.1: (a) a material particle in a body, (b) a place in space, (c) a configuration of the body

By fixing the material particles of the body to points in space, one has a **configuration** of the body $\bar{\chi}$, Fig. 2.1.1c. A configuration can be expressed as a mapping of the particles P to the point $\mathbf{x}$,

$$\mathbf{x} = \bar{\chi}(P) \tag{2.1.1}$$

A **motion** of the body is a *family* of configurations parameterised by time *t*,

$$\mathbf{x} = \bar{\chi}(P, t) \tag{2.1.2}$$

At any time *t*, Eqn. 2.1.2 gives the location in space $\mathbf{x}$ of the material particle P, Fig. 2.1.2.

[1] these particles are not the discrete mass particles of Newtonian mechanics, rather they are very small portions of continuous matter; the meaning of particle is made precise in the definitions which follow

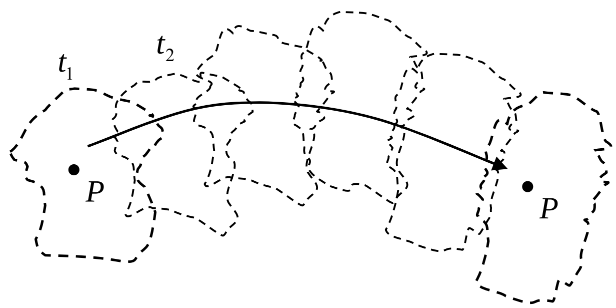

Figure 2.1.2: a motion of material

The Reference and Current Configurations

Choose now some **reference configuration**, Fig. 2.1.3. The motion can then be measured relative to this configuration. The reference configuration might be the configuration occupied by the material at time $t = 0$, in which case it is often called the **initial configuration**. For a solid, it might be natural to choose a configuration for which the material is stress-free, in which case it is often called the **undeformed configuration**. However, the choice of reference configuration is completely arbitrary.

Introduce a Cartesian coordinate system with base vectors $\mathbf{E}_i$ for the reference configuration. A material particle P in the reference configuration can then be assigned a unique position vector $\mathbf{X} = X_i \mathbf{E}_i$ relative to the origin of the axes. The coordinates (X_1, X_2, X_3) of the particle are called **material coordinates** (or **Lagrangian coordinates** or **referential coordinates**).

Some time later, say at time *t*, the material occupies a different configuration, which will be called the **current configuration** (or **deformed configuration**). Introduce a second Cartesian coordinate system with base vectors $\mathbf{e}_i$ for the current configuration, Fig. 2.1.3. In the current configuration, the same particle P now occupies the location $\mathbf{x}$, which can now also be assigned a position vector $\mathbf{x} = x_i \mathbf{e}_i$. The coordinates (x_1, x_2, x_3) are called **spatial coordinates** (or **Eulerian coordinates**).

Each particle thus has two sets of coordinates associated with it. The particle's material coordinates stay with it throughout its motion. The particle's spatial coordinates change as it moves.

The coordinate systems do not have to be Cartesian. For example, suppose one has a rectangular block which deforms into a curved beam (part of a circle). In that case it would be sensible to employ a rectangular Cartesian coordinate system with coordinates (X_1, X_2, X_3) to describe the reference configuration, and a polar coordinate system (r, θ, z) to describe the current configuration.

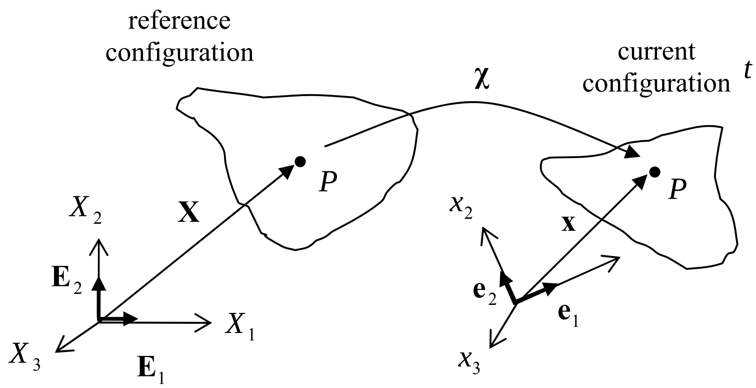

Figure 2.1.3: reference and current configurations

In practice, the material and spatial axes are usually taken to be coincident so that the base vectors $\mathbf{E}_i$ and $\mathbf{e}_i$ are the same, as in Fig. 2.1.4. Nevertheless, the use of different base vectors $\mathbf{E}$ and $\mathbf{e}$ for the reference and current configurations is useful even when the material and spatial axes are coincident, since it helps distinguish between quantities associated with the reference configuration and those associated with the spatial configuration (see later).

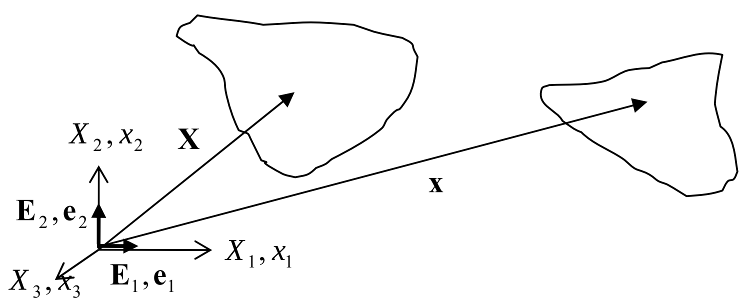

Figure 2.1.4: reference and current configurations with coincident axes

In terms of the position vectors, the motion 2.1.2 can be expressed as a relationship between the material and spatial coordinates,

$$\boxed{\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t), \qquad x_i = \chi_i(X_1, X_2, X_3, t)} \qquad \textbf{Material description} \qquad (2.1.3)$$

or the inverse relation

$$\boxed{\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x},t), \qquad X_i = \chi_i^{-1}(x_1, x_2, x_3, t)} \qquad \textbf{Spatial description} \qquad (2.1.4)$$

If one knows the material coordinates of a particle then its position in the current configuration can be determined from 2.1.3. Alternatively, if one focuses on some location in space, in the current configuration, then the material particle occupying that position can be determined from 2.1.4. This is illustrated in the following example.

Example (Extension of a Bar)

Consider the motion

$$x_1 = 3X_1 t + X_1 + t, \quad x_2 = X_2, \quad x_3 = X_3 \tag{2.1.5}$$

These equations are of the form 2.1.3 and say that "the particle that was originally at position **X** is now, at time *t*, at position **x**". They represent a simple translation and uniaxial extension of material as shown in Fig. 2.1.5. Note that $\mathbf{X} = \mathbf{x}$ at $t = 0$.

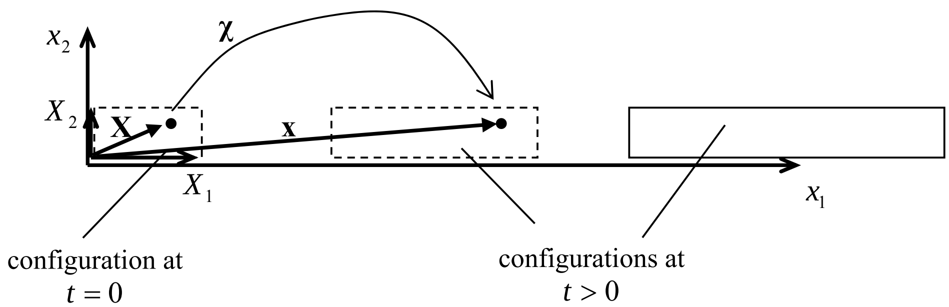

Figure 2.1.5: translation and extension of material

Relations of the form 2.1.4 can be obtained by inverting 2.1.5:

$$X_1 = \frac{x_1 - t}{1 + 3t}, \quad X_2 = x_2, \quad X_3 = x_3$$

These equations say that "the particle that is now, at time *t*, at position **x** was originally at position **X**".

■

Convected Coordinates

The material and spatial coordinate systems used here are fixed Cartesian systems. An alternative method of describing a motion is to *attach* the material coordinate system to the material and let it deform with the material. The motion is then described by defining how this coordinate system changes. This is the **convected coordinate system**. In general, the axes of a convected system will not remain mutually orthogonal and a curvilinear system is required. Convected coordinates will be examined in §2.10.

2.1.2 The Material and Spatial Descriptions

Any physical property (such as density, temperature, etc.) or kinematic property (such as displacement or velocity) of a body can be described in terms of either the material coordinates **X** or the spatial coordinates **x**, since they can be transformed into each other using 2.1.3-4. A **material** (or **Lagrangian**) **description** of events is one where the material coordinates are the independent variables. A **spatial** (or **Eulerian**) description of events is one where the spatial coordinates are used.

Example (Temperature of a Body)

Suppose the temperature θ of a body is, in material coordinates,

$$\theta(\mathbf{X},t) = 3X_1 - X_3 \tag{2.1.6}$$

but, in the spatial description,

$$\theta(\mathbf{x},t) = \frac{x_1}{t} - 1 - x_3 . \tag{2.1.7}$$

According to the material description 2.1.6, the temperature is different for different particles, but the temperature of each particle remains constant over time. The spatial description 2.1.7 describes the time-dependent temperature at a specific location in space, **x**, Fig. 2.1.6. Different material particles are flowing through this location over time.

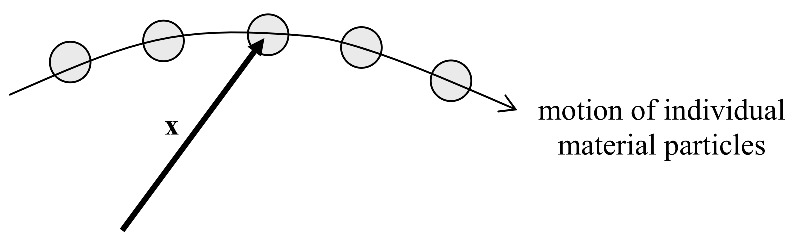

Figure 2.1.6: particles flowing through space

■

In the material description, then, attention is focused on specific *material*. The piece of matter under consideration may change shape, density, velocity, and so on, but it is always the same piece of material. On the other hand, in the spatial description, attention is focused on a fixed location in *space*. Material may pass through this location during the motion, so different material is under consideration at different times.

The spatial description is the one most often used in Fluid Mechanics since there is no natural reference configuration of the material as it is continuously moving. However, both the material and spatial descriptions are used in Solid Mechanics, where the reference configuration is usually the stress-free configuration.

2.1.3 Small Perturbations

A large number of important problems involve materials which deform only by a relatively small amount. An example would be the steel structural columns in a building under modest loading. In this type of problem there is virtually no distinction to be made between the two viewpoints taken above and the analysis is simplified greatly (see later, on Small Strain Theory, §2.7).

2.1.4 Problems

1. The density of a material is given by $\rho = 3X_1 + X_2$ and the motion is given by the equations $X_1 = x_1,\ X_2 = x_2 - t,\ X_3 = x_3 - t$.
   (a) what kind of description is this for the density, and what kind of description is this for the motion?
   (b) re-write the density in terms of **x** – what is the name given to this description of the density?
   (c) is the density of any given material particle changing with time?
   (d) invert the motion equations so that **X** is the independent variable – what is the name given to this description of the motion?
   (e) draw the line element joining the origin to $(1,1,0)$ and sketch the position of this element of material at times $t = 1$ and $t = 2$.

2.2 Deformation and Strain

A number of useful ways of describing and quantifying the deformation of a material are discussed in this section.

Attention is restricted to the reference and current configurations. No consideration is given to the particular sequence by which the current configuration is reached from the reference configuration and so the deformation can be considered to be independent of time. In what follows, particles in the reference configuration will often be termed "undeformed" and those in the current configuration "deformed".

In a change from Chapter 1, lower case letters will now be reserved for both vector- *and* tensor- functions of the spatial coordinates **x**, whereas upper-case letters will be reserved for functions of material coordinates **X**. There will be exceptions to this, but it should be clear from the context what is implied.

2.2.1 The Deformation Gradient

The **deformation gradient F** is the fundamental measure of deformation in continuum mechanics. It is the second order tensor which maps line elements in the reference configuration into line elements (consisting of the *same* material particles) in the current configuration.

Consider a line element $d\mathbf{X}$ emanating from position **X** in the reference configuration which becomes $d\mathbf{x}$ in the current configuration, Fig. 2.2.1. Then, using 2.1.3,

$$\begin{aligned} d\mathbf{x} &= \boldsymbol{\chi}(\mathbf{X} + d\mathbf{X}) - \boldsymbol{\chi}(\mathbf{X}) \\ &= (\operatorname{Grad} \boldsymbol{\chi}) d\mathbf{X} \end{aligned} \tag{2.2.1}$$

A capital G is used on "Grad" to emphasise that this is a gradient with respect to the material coordinates[1], the **material gradient**, $\partial \boldsymbol{\chi} / \partial \mathbf{X}$.

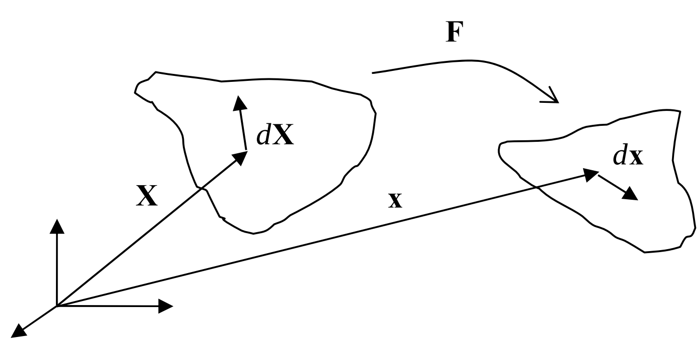

Figure 2.2.1: the Deformation Gradient acting on a line element

[1] one can have material gradients and spatial gradients of material or spatial fields – see later

The motion vector-function $\boldsymbol{\chi}$ in 2.1.3, 2.2.1, is a function of the variable **X**, but it is customary to denote this simply by **x**, the value of $\boldsymbol{\chi}$ at **X**, i.e. $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, so that

$$\boxed{\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \operatorname{Grad} \mathbf{x}, \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}} \quad \textbf{Deformation Gradient} \qquad (2.2.2)$$

with

$$\boxed{d\mathbf{x} = \mathbf{F}\, d\mathbf{X}, \qquad dx_i = F_{iJ}\, dX_J} \quad \textbf{action of F} \qquad (2.2.3)$$

Lower case indices are used in the index notation to denote quantities associated with the spatial basis $\{\mathbf{e}_i\}$ whereas upper case indices are used for quantities associated with the material basis $\{\mathbf{E}_I\}$.

Note that

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X}$$

is a differential quantity and this expression has some error associated with it; the error (due to terms of order $(d\mathbf{X})^2$ and higher, neglected from a Taylor series) tends to zero as the differential $d\mathbf{X} \to 0$. The deformation gradient (whose components are finite) thus characterises the deformation in the *neighbourhood* of a point **X**, mapping infinitesimal line elements $d\mathbf{X}$ emanating from **X** in the reference configuration to the infinitesimal line elements $d\mathbf{x}$ emanating from **x** in the current configuration, Fig. 2.2.2.

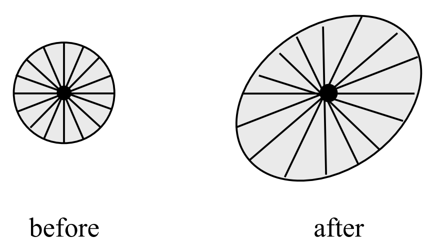

Figure 2.2.2: deformation of a material particle

Example

Consider the cube of material with sides of unit length illustrated by dotted lines in Fig. 2.2.3. It is deformed into the rectangular prism illustrated (this could be achieved, for example, by a continuous rotation and stretching motion). The material and spatial coordinate axes are coincident. The material description of the deformation is

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = -6X_2\mathbf{e}_1 + \frac{1}{2}X_1\mathbf{e}_2 + \frac{1}{3}X_3\mathbf{e}_3$$

and the spatial description is

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = 2x_2\mathbf{E}_1 - \frac{1}{6}x_1\mathbf{E}_2 + 3x_3\mathbf{E}_3$$

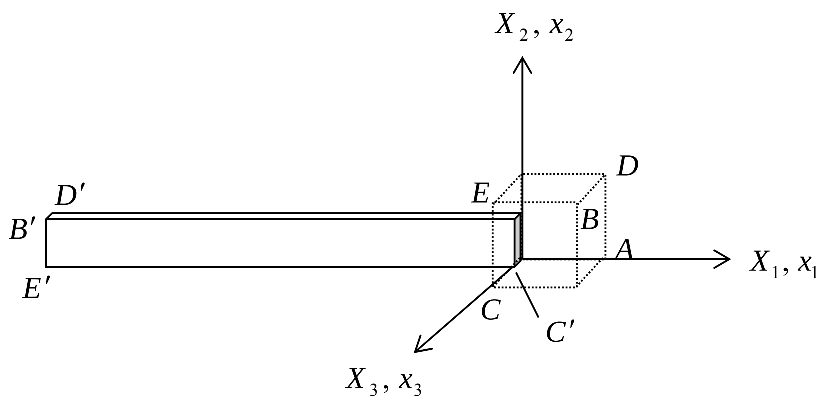

Figure 2.2.3: a deforming cube

Then

$$\mathbf{F} = \frac{\partial x_i}{\partial X_J} = \begin{bmatrix} 0 & -6 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Once **F** is known, the position of any element can be determined. For example, taking a line element $d\mathbf{X} = [da, 0, 0]^{\mathrm{T}}$, $d\mathbf{x} = \mathbf{F}d\mathbf{X} = [0, da/2, 0]^{\mathrm{T}}$.

■

Homogeneous Deformations

A **homogeneous deformation** is one where the deformation gradient is uniform, i.e. independent of the coordinates, and the associated motion is termed **affine**. Every part of the material deforms as the whole does, and straight parallel lines in the reference configuration map to straight parallel lines in the current configuration, as in the above example. Most examples to be considered in what follows will be of homogeneous deformations; this keeps the algebra to a minimum, but homogeneous deformation analysis is very useful in itself since most of the basic experimental testing of materials, e.g. the uniaxial tensile test, involve homogeneous deformations.

Rigid Body Rotations and Translations

One can add a constant vector **c** to the motion, $\mathbf{x} = \mathbf{x} + \mathbf{c}$, without changing the deformation, $\mathrm{Grad}(\mathbf{x} + \mathbf{c}) = \mathrm{Grad}\mathbf{x}$. Thus the deformation gradient does not take into account rigid-body **translations** of bodies in space. If a body only translates as a rigid body in space, then $\mathbf{F} = \mathbf{I}$, and $\mathbf{x} = \mathbf{X} + \mathbf{c}$ (again, note that **F** does not tell us where in space a particle is, only how it has deformed locally). If there is *no* motion, then not only is $\mathbf{F} = \mathbf{I}$, but $\mathbf{x} = \mathbf{X}$.

If the body rotates as a rigid body (with no translation), then $\mathbf{F} = \mathbf{R}$, a rotation tensor (§1.10.8). For example, for a rotation of θ about the X_2 axis,

$$\mathbf{F} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}$$

Note that different particles of the same material body can be translating only, rotating only, deforming only, or any combination of these.

The Inverse of the Deformation Gradient

The inverse deformation gradient $\mathbf{F}^{-1}$ carries the spatial line element d**x** to the material line element d**X**. It is defined as

$$\boxed{\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \operatorname{grad}\mathbf{X}, \qquad F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j}} \qquad \textbf{Inverse Deformation Gradient} \qquad (2.2.4)$$

so that

$$\boxed{d\mathbf{X} = \mathbf{F}^{-1}\, d\mathbf{x}, \qquad dX_I = F_{Ij}^{-1}\, dx_j} \qquad \textbf{action of } \mathbf{F}^{-1} \qquad (2.2.5)$$

with (see Eqn. 1.15.2)

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{F}\mathbf{F}^{-1} = \mathbf{I} \qquad F_{iM}F_{Mj}^{-1} = \delta_{ij} \qquad (2.2.6)$$

Cartesian Base Vectors

Explicitly, in terms of the material and spatial base vectors (see 1.14.3),

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial X_J} \otimes \mathbf{E}_J = \frac{\partial x_i}{\partial X_J} \mathbf{e}_i \otimes \mathbf{E}_J \\ \mathbf{F}^{-1} &= \frac{\partial \mathbf{X}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial X_I}{\partial x_j} \mathbf{E}_I \otimes \mathbf{e}_j \end{aligned} \qquad (2.2.7)$$

so that, for example, $\mathbf{F}d\mathbf{X} = \left(\partial x_i / \partial X_J\right)\mathbf{e}_i \otimes \mathbf{E}_J\left(dX_M \mathbf{E}_M\right) = \left(\partial x_i / \partial X_J\right) dX_J \mathbf{e}_i = d\mathbf{x}$.

Because **F** and $\mathbf{F}^{-1}$ act on vectors in one configuration to produce vectors in the other configuration, they are termed **two-point tensors**. They are defined in both configurations. This is highlighted by their having both reference and current base vectors **E** and **e** in their Cartesian representation 2.2.7.

Here follow some important relations which relate scalar-, vector- and second-order tensor-valued functions in the material and spatial descriptions, the first two relating the material and spatial gradients {▲Problem 1}.

$$\begin{aligned} \mathrm{grad}\phi &= \mathrm{Grad}\phi\,\mathbf{F}^{-1} \\ \mathrm{grad}\mathbf{v} &= \mathrm{Grad}\mathbf{V}\,\mathbf{F}^{-1} \\ \mathrm{div}\mathbf{a} &= \mathrm{Grad}\mathbf{A} : \mathbf{F}^{-\mathrm{T}} \end{aligned} \tag{2.2.8}$$

Here, ϕ is a scalar; **V** and **v** are the *same* vector, the former being a function of the material coordinates, the material description, the latter a function of the spatial coordinates, the spatial description. Similarly, **A** is a second order tensor in the material form and **a** is the equivalent spatial form.

The first two of 2.2.8 relate the material gradient to the spatial gradient: the gradient of a function is a measure of how the function changes as one moves through space; since the material coordinates and the spatial coordinates differ, the change in a function with respect to a unit change in the material coordinates will differ from the change in the *same* function with respect to a unit change in the spatial coordinates (see also §2.2.7 below).

Example

Consider the deformation

$$\begin{aligned} \mathbf{x} &= (2X_2 - X_3)\mathbf{e}_1 + (-X_2)\mathbf{e}_2 + (X_1 + 3X_2 + X_3)\mathbf{e}_3 \\ \mathbf{X} &= (x_1 + 5x_2 + x_3)\mathbf{E}_1 + (-x_2)\mathbf{E}_2 + (-x_1 - 2x_2)\mathbf{E}_3 \end{aligned}$$

so that

$$\mathbf{F} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ 0 & -1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$

Consider the vector $\mathbf{v}(\mathbf{x}) = (2x_1 - x_2)\mathbf{e}_1 + (-3x_2^2 + x_3)\mathbf{e}_2 + (x_1 + x_3)\mathbf{e}_3$ which, in the material description, is

$$\mathbf{V}(\mathbf{X}) = (5X_2 - 2X_3)\mathbf{E}_1 + (X_1 + 3X_2 + X_3 - 3X_2^2)\mathbf{E}_2 + (X_1 + 5X_2)\mathbf{E}_3$$

The material and spatial gradients are

$$\mathrm{Grad}\mathbf{V} = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 3 - 6X_2 & 1 \\ 1 & 5 & 0 \end{bmatrix}, \qquad \mathrm{grad}\mathbf{v} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and it can be seen that

$$\text{Grad}\mathbf{V}\,\mathbf{F}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 6X_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \text{grad}\,\mathbf{v}$$

■

2.2.2 The Cauchy-Green Strain Tensors

The deformation gradient describes how a line element in the reference configuration maps into a line element in the current configuration. It has been seen that the deformation gradient gives information about deformation (change of shape) and rigid body rotation, but does not encompass information about possible rigid body translations. The deformation and rigid rotation will be separated shortly (see §2.2.5). To this end, consider the following **strain** tensors; these tensors give direct information about the deformation of the body. Specifically, the **Left Cauchy-Green Strain** and **Right Cauchy-Green Strain** tensors give a measure of how the lengths of line elements and angles between line elements (through the vector dot product) change between configurations.

The Right Cauchy-Green Strain

Consider two line elements in the reference configuration $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$ which are mapped into the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$ in the current configuration. Then, using 1.10.3d,

$$\boxed{\begin{aligned} d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} &= \left(\mathbf{F}d\mathbf{X}^{(1)}\right)\cdot\left(\mathbf{F}d\mathbf{X}^{(2)}\right) \\ &= d\mathbf{X}^{(1)}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)}\mathbf{C}\,d\mathbf{X}^{(2)} \end{aligned}} \qquad \textbf{action of C} \qquad (2.2.9)$$

where, by definition, $\mathbf{C}$ is the right Cauchy-Green Strain[2]

$$\boxed{\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \qquad C_{IJ} = F_{kI}F_{kJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}} \qquad \textbf{Right Cauchy-Green Strain} \qquad (2.2.10)$$

It is a symmetric, positive definite (which will be clear from Eqn. 2.2.17 below), tensor, which implies that it has real positive eigenvalues (*cf*. §1.11.2), and this has important consequences (see later). Explicitly in terms of the base vectors,

$$\mathbf{C} = \left(\frac{\partial x_k}{\partial X_I}\mathbf{E}_I \otimes \mathbf{e}_k\right)\left(\frac{\partial x_m}{\partial X_J}\mathbf{e}_m \otimes \mathbf{E}_J\right) = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}\mathbf{E}_I \otimes \mathbf{E}_J\,. \qquad (2.2.11)$$

Just as the line element $d\mathbf{X}$ is a vector defined in and associated with the reference configuration, $\mathbf{C}$ is defined in and associated with the reference configuration, acting on vectors in the reference configuration, and so is called a **material tensor**.

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[2] "right" because **F** is on the right of the formula

The inverse of **C**, $\mathbf{C}^{-1}$, is called the **Piola deformation tensor**.

The Left Cauchy-Green Strain

Consider now the following, using Eqn. 1.10.18c:

$$\boxed{\begin{aligned} d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} &= \left(\mathbf{F}^{-1} d\mathbf{x}^{(1)}\right) \cdot \left(\mathbf{F}^{-1} d\mathbf{x}^{(2)}\right) \\ &= d\mathbf{x}^{(1)} \left(\mathbf{F}^{-\mathrm{T}} \mathbf{F}^{-1}\right) d\mathbf{x}^{(2)} \\ &= d\mathbf{x}^{(1)} \mathbf{b}^{-1} d\mathbf{x}^{(2)} \end{aligned}} \quad \textbf{action of } \mathbf{b}^{-1} \qquad (2.2.12)$$

where, by definition, **b** is the left Cauchy-Green Strain, also known as the **Finger tensor**:

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \qquad b_{ij} = F_{iK} F_{jK} = \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_K}} \quad \textbf{Left Cauchy-Green Strain} \qquad (2.2.13)$$

Again, this is a symmetric, positive definite tensor, only here, **b** is defined in the current configuration and so is called a **spatial tensor**.

The inverse of **b**, $\mathbf{b}^{-1}$, is called the **Cauchy deformation tensor**.

It can be seen that the right and left Cauchy-Green tensors are related through

$$\mathbf{C} = \mathbf{F}^{-1}\mathbf{b}\mathbf{F}, \qquad \mathbf{b} = \mathbf{F}\mathbf{C}\mathbf{F}^{-1} \qquad (2.2.14)$$

Note that tensors can be material (e.g. **C**), two-point (e.g. **F**) or spatial (e.g. **b**). Whatever type they are, they can always be described using material or spatial coordinates through the motion mapping 2.1.3, that is, using the material or spatial descriptions. Thus one distinguishes between, for example, a spatial tensor, which is an intrinsic property of a tensor, and the spatial description of a tensor.

The Principal Scalar Invariants of the Cauchy-Green Tensors

Using 1.10.10b,

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right) = \mathrm{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = \mathrm{tr}\mathbf{b} \qquad (2.2.15)$$

This holds also for arbitrary powers of these tensors, $\mathrm{tr}\mathbf{C}^n = \mathrm{tr}\mathbf{b}^n$, and therefore, from Eqn. 1.11.17, the invariants of **C** and **b** are equal.

2.2.3 The Stretch

The **stretch** (or the **stretch ratio**) λ is defined as the ratio of the length of a deformed line element to the length of the corresponding undeformed line element:

$$\boxed{\lambda = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}} \qquad \textbf{The Stretch} \tag{2.2.16}$$

From the relations involving the Cauchy-Green Strains, letting $d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} \equiv d\mathbf{X}$, $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)} \equiv d\mathbf{x}$, and dividing across by the square of the length of $d\mathbf{X}$ or $d\mathbf{x}$,

$$\lambda^2 = \left(\frac{|d\mathbf{x}|}{|d\mathbf{X}|}\right)^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}, \qquad \lambda^{-2} = \left(\frac{|d\mathbf{X}|}{|d\mathbf{x}|}\right)^2 = d\hat{\mathbf{x}}\mathbf{b}^{-1}d\hat{\mathbf{x}} \tag{2.2.17}$$

Here, the quantities $d\hat{\mathbf{X}} = d\mathbf{X}/|d\mathbf{X}|$ and $d\hat{\mathbf{x}} = d\mathbf{x}/|d\mathbf{x}|$ are unit vectors in the directions of $d\mathbf{X}$ and $d\mathbf{x}$. Thus, through these relations, **C** and **b** determine how much a line element stretches (and, from 2.2.17, **C** and **b** can be seen to be indeed positive definite).

One says that a line element is **extended**, **unstretched** or **compressed** according to $\lambda > 1$, $\lambda = 1$ or $\lambda < 1$.

Stretching along the Coordinate Axes

Consider three line elements lying along the three coordinate axes[3]. Suppose that the material deforms in a special way, such that these line elements undergo a **pure stretch**, that is, they change length with no change in the right angles between them. If the stretches in these directions are λ_1, λ_2 and λ_3, then

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3 \tag{2.2.18}$$

and the deformation gradient has only diagonal elements in its matrix form:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad F_{iJ} = \lambda_i \delta_{iJ} \quad \text{(no sum)} \tag{2.2.19}$$

Whereas material undergoes pure stretch along the coordinate directions, line elements off-axes will in general stretch/contract *and* rotate relative to each other. For example, a line element $d\mathbf{X} = [\alpha, \alpha, 0]^{\mathrm{T}}$ stretches by $\lambda = \sqrt{d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}} = \sqrt{\left(\lambda_1^2 + \lambda_2^2\right)/2}$ with $d\mathbf{x} = [\lambda_1\alpha, \lambda_2\alpha, 0]^{\mathrm{T}}$, and rotates if $\lambda_1 \neq \lambda_2$.

It will be shown below that, for any deformation, there are always three mutually orthogonal directions along which material undergoes a pure stretch. These directions, the coordinate axes in this example, are called the **principal axes** of the material and the associated stretches are called the **principal stretches**.

[3] with the material and spatial basis vectors coincident

The Case of F Real and Symmetric

Consider now another special deformation, where **F** is a real symmetric tensor, in which case the eigenvalues are real and the eigenvectors form an orthonormal basis (*cf*. §1.11.2)[4]. In any given coordinate system, **F** will in general result in the stretching of line elements and the changing of the angles between line elements. However, if one chooses a coordinate set to be the eigenvectors of **F**, then from Eqn. 1.11.11-12 one can write[5]

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad [\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.2.20}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of **F**. The eigenvalues are the principal stretches and the eigenvectors are the principal axes. This indicates that as long as **F** is real and symmetric, one can always find a coordinate system along whose axes the material undergoes a pure stretch, with no rotation. This topic will be discussed more fully in §2.2.5 below.

2.2.4 The Green-Lagrange and Euler-Almansi Strain Tensors

Whereas the left and right Cauchy-Green tensors give information about the change in angle between line elements and the stretch of line elements, the **Green-Lagrange strain** and the **Euler-Almansi strain** tensors directly give information about the change in the squared length of elements.

Specifically, when the Green-Lagrange strain **E** operates on a line element d**X**, it gives (half) the change in the squares of the undeformed and deformed lengths:

$$\begin{aligned} \frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} &= \frac{1}{2}\{d\mathbf{X}\mathbf{C}d\mathbf{X} - d\mathbf{X}\cdot d\mathbf{X}\} \\ &= \frac{1}{2}\{d\mathbf{X}(\mathbf{C}-\mathbf{I})d\mathbf{X}\} \\ &\equiv d\mathbf{X}\mathbf{E}d\mathbf{X} \end{aligned} \qquad \textbf{action of E} \tag{2.2.21}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}), \quad E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \qquad \textbf{Green-Lagrange Strain} \tag{2.2.22}$$

It is a symmetric positive definite material tensor. Similarly, the (symmetric spatial) Euler-Almansi strain tensor is defined through

---

[4] in fact, **F** in this case will have to be positive definite, with $\det \mathbf{F} > 0$ (see later in §2.2.8)

[5] $\hat{\mathbf{n}}_i$ are the eigenvectors for the basis $\mathbf{e}_i$, $\hat{\mathbf{N}}_I$ for the basis $\hat{\mathbf{E}}_i$, with $\hat{\mathbf{n}}_i$, $\hat{\mathbf{N}}_I$ coincident; when the bases are not coincident, the notion of rotating line elements becomes ambiguous – this topic will be examined later in the context of *objectivity*

$$\boxed{\frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} = d\mathbf{x}\,\mathbf{e}\,d\mathbf{x}} \qquad \textbf{action of e} \qquad (2.2.23)$$

and

$$\boxed{\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right)} \qquad \textbf{Euler-Almansi Strain} \qquad (2.2.24)$$

Physical Meaning of the Components of E

Take a line element in the 1-direction, $d\mathbf{X}_{(1)} = [dX_1,\ 0,\ 0]^{\mathrm{T}}$, so that $d\hat{\mathbf{X}}_{(1)} = [1,\ 0,\ 0]^{\mathrm{T}}$. The square of the stretch of this element is

$$\lambda_{(1)}^2 = d\hat{\mathbf{X}}_{(1)}\mathbf{C}\,d\hat{\mathbf{X}}_{(1)} = C_{11} \quad \rightarrow \quad E_{11} = \frac{1}{2}\left(C_{11} - 1\right) = \frac{1}{2}\left(\lambda_{(1)}^2 - 1\right)$$

The unit extension is $\left(|d\mathbf{x}| - |d\mathbf{X}|\right)/|d\mathbf{X}| = \lambda - 1$. Denoting the unit extension of $d\mathbf{X}_{(1)}$ by $\mathbf{E}_{(1)}$, one has

$$E_{11} = \mathbf{E}_{(1)} + \frac{1}{2}\mathbf{E}_{(1)}^2 \qquad (2.2.25)$$

and similarly for the other diagonal elements E_{22}, E_{33}.

When the deformation is small, $\mathbf{E}_{(1)}^2$ is small in comparison to $\mathbf{E}_{(1)}$, so that $E_{11} \approx \mathbf{E}_{(1)}$. For small deformations then, the diagonal terms are equivalent to the unit extensions.

Let θ_{12} denote the angle between the deformed elements which were initially parallel to the X_1 and X_2 axes. Then

$$\begin{aligned}\cos\theta_{12} &= \frac{d\mathbf{x}_{(1)}}{|d\mathbf{x}_{(1)}|} \cdot \frac{d\mathbf{x}_{(2)}}{|d\mathbf{x}_{(2)}|} = \frac{|d\mathbf{X}_{(1)}||d\mathbf{X}_{(2)}|}{|d\mathbf{x}_{(1)}||d\mathbf{x}_{(2)}|}\left\{\frac{d\mathbf{X}_{(1)}}{|d\mathbf{X}_{(1)}|} \cdot \mathbf{C}\frac{d\mathbf{X}_{(2)}}{|d\mathbf{X}_{(2)}|}\right\} = \frac{C_{12}}{\lambda_{(1)}\lambda_{(2)}} \\ &= \frac{2E_{12}}{\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}}\end{aligned} \qquad (2.2.26)$$

and similarly for the other off-diagonal elements. Note that if $\theta_{12} = \pi/2$, so that there is no angle change, then $E_{12} = 0$. Again, if the deformation is small, then E_{11}, E_{22} are small, and

$$\frac{\pi}{2} - \theta_{12} \approx \sin\left(\frac{\pi}{2} - \theta_{12}\right) = \cos\theta_{12} \approx 2E_{12} \qquad (2.2.27)$$

In words: for small deformations, the component E_{12} gives half the change in the original right angle.

2.2.5 Stretch and Rotation Tensors

The deformation gradient can always be decomposed into the product of two tensors, a stretch tensor and a rotation tensor (in one of two different ways, material or spatial versions). This is known as the **polar decomposition**, and is discussed in §1.11.7. One has

$$\boxed{\mathbf{F} = \mathbf{R}\mathbf{U}} \qquad \textbf{Polar Decomposition (Material)} \qquad (2.2.28)$$

Here, **R** is a proper orthogonal tensor, i.e. $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ with $\det \mathbf{R} = 1$, called *the* **rotation tensor**. It is a measure of the local rotation at **X**.

The decomposition is not unique; it is made unique by choosing **U** to be a *symmetric* tensor, called the **right stretch tensor**. It is a measure of the local stretching (or contraction) of material at **X**. Consider a line element $d\mathbf{X}$. Then

$$\lambda d\hat{\mathbf{x}} = \mathbf{F} d\hat{\mathbf{X}} = \mathbf{R}\mathbf{U} d\hat{\mathbf{X}} \qquad (2.2.29)$$

and so {▲Problem 2}

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{U} \cdot \mathbf{U} d\hat{\mathbf{X}} \qquad (2.2.30)$$

Thus (this is a definition of **U**)

$$\boxed{\mathbf{U} = \sqrt{\mathbf{C}} \qquad (\mathbf{C} = \mathbf{U}\mathbf{U})} \qquad \textbf{The Right Stretch Tensor} \qquad (2.2.31)$$

From 2.2.30, the right Cauchy-Green strain **C** (and by consequence the Euler-Lagrange strain **E**) only give information about the stretch of line elements; it does not give information about the rotation that is experienced by a particle during motion. The deformation gradient **F**, however, contains information about both the stretch and rotation. It can also be seen from 2.2.30-1 that **U** is a material tensor.

Note that, since

$$d\mathbf{x} = \mathbf{R}(\mathbf{U} d\mathbf{X}),$$

the undeformed line element is *first* stretched by **U** and is *then* rotated by **R** into the deformed element $d\mathbf{x}$ (the element may also undergo a rigid body translation **c**), Fig. 2.2.4. **R** is a two-point tensor.

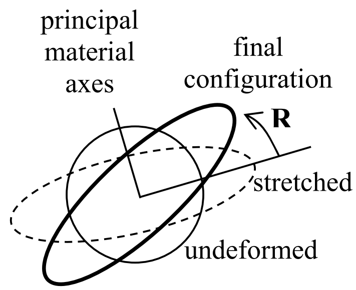


Figure 2.2.4: the polar decomposition

Evaluation of U

In order to evaluate **U**, it is necessary to evaluate $\sqrt{\mathbf{C}}$. To evaluate the square-root, **C** must first be obtained in relation to its principal axes, so that it is diagonal, and then the square root can be taken of the diagonal elements, since its eigenvalues will be positive (see §1.11.6). Then the tensor needs to be transformed back to the original coordinate system.

Example

Consider the motion

$$x_1 = 2X_1 - 2X_2, \quad x_2 = X_1 + X_2, \quad x_3 = X_3$$

The (homogeneous) deformation of a unit square in the $x_1 - x_2$ plane is as shown in Fig. 2.2.5.

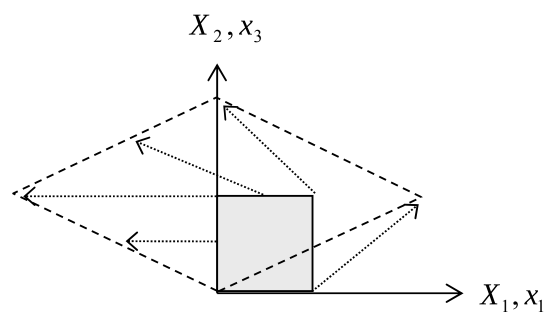


Figure 2.2.5: deformation of a square

One has

$$[\mathbf{F}] = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : (\mathbf{e}_i \otimes \mathbf{E}_j), \qquad [\mathbf{C}] = [\mathbf{F}^{\mathrm{T}}\mathbf{F}] = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : (\mathbf{E}_i \otimes \mathbf{E}_j)$$

Note that **F** is not symmetric, so that it might have only one real eigenvalue (in fact here it does have complex eigenvalues), and the eigenvectors may not be orthonormal. **C**, on the other hand, by its very definition, *is* symmetric; it is in fact positive definite and so has positive real eigenvalues forming an orthonormal set.

To determine the principal axes of **C**, it is necessary to evaluate the eigenvalues/eigenvectors of the tensor. The eigenvalues are the roots of the characteristic equation 1.11.5,

$$\alpha^3 - \mathrm{I}_{\mathbf{C}}\alpha^2 + \mathrm{II}_{\mathbf{C}}\alpha - \mathrm{III}_{\mathbf{C}} = 0$$

and the first, second and third invariants of the tensor are given by 1.11.6 so that $\alpha^3 - 11\alpha^2 + 26\alpha - 16 = 0$, with roots $\alpha = 8, 2, 1$. The three corresponding eigenvectors are found from 1.11.8,

$$\begin{aligned} (C_{11}-\alpha)\hat{N}_1 + C_{12}\hat{N}_2 + C_{13}\hat{N}_3 &= 0 \\ C_{21}\hat{N}_1 + (C_{22}-\alpha)\hat{N}_2 + C_{23}\hat{N}_3 &= 0 \\ C_{31}\hat{N}_1 + C_{32}\hat{N}_2 + (C_{33}-\alpha)\hat{N}_3 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} (5-\alpha)\hat{N}_1 - 3\hat{N}_2 &= 0 \\ -3\hat{N}_1 + (5-\alpha)\hat{N}_2 &= 0 \\ (1-\alpha)\hat{N}_3 &= 0 \end{aligned}$$

Thus (normalizing the eigenvectors so that they are unit vectors, and form a right-handed set, Fig. 2.2.6):

(i) for $\alpha = 8$, $-3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 - 3\hat{N}_2 = 0, -7\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 - \frac{1}{\sqrt{2}}\mathbf{E}_2$

(ii) for $\alpha = 2$, $3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 3\hat{N}_2 = 0, -\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_2 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$

(iii) for $\alpha = 1$, $4\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 4\hat{N}_2 = 0, 0\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$

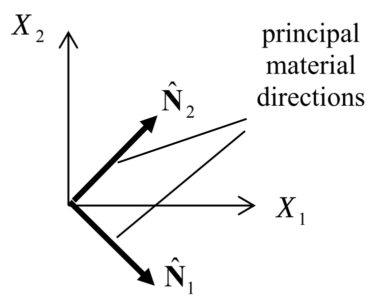


Figure 2.2.6: deformation of a square

Thus the right Cauchy-Green strain tensor **C**, with respect to coordinates with base vectors $\mathbf{E}'_1 = \hat{\mathbf{N}}_1$, $\mathbf{E}'_2 = \hat{\mathbf{N}}_2$ and $\mathbf{E}'_3 = \hat{\mathbf{N}}_3$, that is, in terms of principal coordinates, is

$$[\mathbf{C}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

This result can be checked using the tensor transformation formulae 1.13.6, $[\mathbf{C}'] = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{C}][\mathbf{Q}]$, where Q is the transformation matrix of direction cosines (see also the example at the end of §1.5.2),

$$Q_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1' & \mathbf{e}_1 \cdot \mathbf{e}_2' & \mathbf{e}_1 \cdot \mathbf{e}_3' \\ \mathbf{e}_2 \cdot \mathbf{e}_1' & \mathbf{e}_2 \cdot \mathbf{e}_2' & \mathbf{e}_2 \cdot \mathbf{e}_3' \\ \mathbf{e}_3 \cdot \mathbf{e}_1' & \mathbf{e}_3 \cdot \mathbf{e}_2' & \mathbf{e}_3 \cdot \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 & \hat{\mathbf{N}}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stretch tensor **U**, with respect to the principal directions is

$$[\mathbf{U}] = \left[\sqrt{\mathbf{C}}\right] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{basis} : \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

These eigenvalues of **U** (which are the square root of those of **C**) are the principal stretches and, as before, they are labeled $\lambda_1, \lambda_2, \lambda_3$.

In the original coordinate system, using the inverse tensor transformation rule 1.13.6, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$,

$$[\mathbf{U}] = \begin{bmatrix} 3/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 3/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \mathbf{E}_i \otimes \mathbf{E}_j$$

so that

$$[\mathbf{R}] = \left[\mathbf{F}\mathbf{U}^{-1}\right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \mathbf{e}_i \otimes \mathbf{E}_j$$

and it can be verified that **R** is a rotation tensor, i.e. is proper orthogonal.

Returning to the deformation of the unit square, the stretch and rotation are as illustrated in Fig. 2.2.7 – the action of **U** is indicated by the arrows, deforming the unit square to the dotted parallelogram, whereas **R** rotates the parallelogram through 45° as a rigid body to its final position.

Note that the line elements along the diagonals (indicated by the heavy lines) lie along the principal directions of **U** and therefore undergo a pure stretch; the diagonal in the $\hat{\mathbf{N}}_1$ direction has stretched but has also moved with a rigid translation.

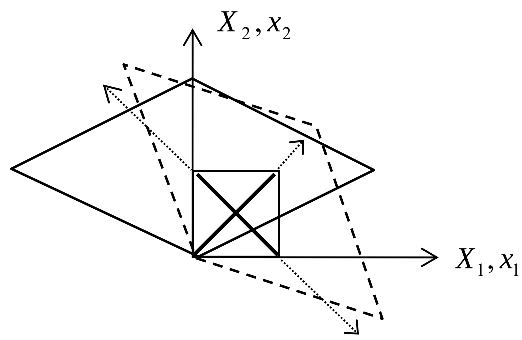


Figure 2.2.7: stretch and rotation of a square

■

Spatial Description

A polar decomposition can be made in the spatial description. In that case,

$$\boxed{\mathbf{F} = \mathbf{vR}} \qquad \textbf{Polar Decomposition (Spatial)} \tag{2.2.32}$$

Here **v** is a symmetric, positive definite second order tensor called the **left stretch tensor**, and $\mathbf{vv} = \mathbf{b}$, where **b** is the left Cauchy-Green tensor. **R** is the same rotation tensor as appears in the material description. Thus an elemental sphere can be regarded as first stretching into an ellipsoid, whose axes are the principal material axes (the principal axes of **U**), and then rotating; or first rotating, and then stretching into an ellipsoid whose axes are the **principal spatial axes** (the principal axes of **v**). The end result is the same.

The development in the spatial description is similar to that given above for the material description, and one finds by analogy with 2.2.30,

$$\lambda^{-2} = d\hat{\mathbf{x}}\mathbf{v}^{-1} \cdot \mathbf{v}^{-1} d\hat{\mathbf{x}} \tag{2.2.33}$$

In the above example, it turns out that **v** takes the simple diagonal form

$$[\mathbf{v}] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \mathbf{e}_i \otimes \mathbf{e}_j .$$

so the unit square rotates first and then undergoes a pure stretch along the coordinate axes, which are the principal spatial axes, and the sequence is now as shown in Fig. 2.2.9.

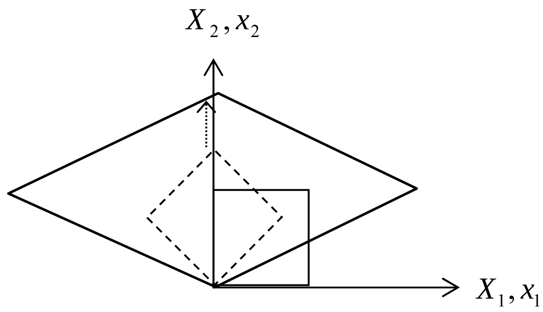


Figure 2.2.8: stretch and rotation of a square in spatial description

Relationship between the Material and Spatial Decompositions

Comparing the two decompositions, one sees that the material and spatial tensors involved are related through

$$\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{b} = \mathbf{R}\mathbf{C}\mathbf{R}^{\mathrm{T}} \tag{2.2.34}$$

Further, suppose that **U** has an eigenvalue λ and an eigenvector $\hat{\mathbf{N}}$. Then $\mathbf{U}\hat{\mathbf{N}} = \lambda\hat{\mathbf{N}}$, so that $\mathbf{RUN} = \lambda\mathbf{RN}$. But $\mathbf{RU} = \mathbf{vR}$, so $\mathbf{v}(\mathbf{R}\hat{\mathbf{N}}) = \lambda(\mathbf{R}\hat{\mathbf{N}})$. Thus **v** also has an eigenvalue λ, but an eigenvector $\hat{\mathbf{n}} = \mathbf{R}\hat{\mathbf{N}}$. From this, it is seen that the rotation tensor **R** maps the principal material axes into the principal spatial axes. It also follows that **R** and **F** can be written explicitly in terms of the material and spatial principal axes (compare the first of these with 1.10.25)[6]:

$$\mathbf{R} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad \mathbf{F} = \mathbf{RU} = \mathbf{R}\sum_{i=1}^{3}\lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \sum_{i=1}^{3}\lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i \tag{2.2.35}$$

and the deformation gradient acts on the principal axes base vectors according to {▲Problem 4}

$$\mathbf{F}\hat{\mathbf{N}}_i = \lambda_i \hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-\mathrm{T}}\hat{\mathbf{N}}_i = \frac{1}{\lambda_i}\hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-1}\hat{\mathbf{n}}_i = \frac{1}{\lambda_i}\hat{\mathbf{N}}_i, \qquad \mathbf{F}^{\mathrm{T}}\hat{\mathbf{n}}_i = \lambda_i \hat{\mathbf{N}}_i \tag{2.2.36}$$

The representation of **F** and **R** in terms of both material and spatial principal base vectors in 2.3.35 highlights their two-point character.

Other Strain Measures

Some other useful measures of strain are

The **Hencky strain** measure: $\mathbf{H} \equiv \ln \mathbf{U}$ (material) or $\mathbf{h} = \ln \mathbf{v}$ (spatial)

[6] this is not a spectral decomposition of **F** (unless **F** happens to be symmetric, which it must be in order to have a spectral decomposition)

The **Biot strain** measure: $\overline{\mathbf{B}} = \mathbf{U} - \mathbf{I}$ (material) or $\overline{\mathbf{b}} = \mathbf{v} - \mathbf{I}$ (spatial)

The Hencky strain is evaluated by first evaluating **U** along the principal axes, so that the logarithm can be taken of the diagonal elements.

The material tensors **H**, $\overline{\mathbf{B}}$, **C**, **U** and **E** are coaxial tensors, with the same eigenvectors $\hat{\mathbf{N}}_i$. Similarly, the spatial tensors **h**, $\overline{\mathbf{b}}$, **b**, **v** and **e** are coaxial with the same eigenvectors $\hat{\mathbf{n}}_i$. From the definitions, the spectral decompositions of these tensors are

$$
\begin{aligned}
\mathbf{U} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{v} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{C} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{b} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{E} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(\lambda_i^2 - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{e} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(1 - 1/\lambda_i^2\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{H} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{h} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\overline{\mathbf{B}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \overline{\mathbf{b}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i
\end{aligned}
\tag{2.2.37}
$$

Deformation of a Circular Material Element

A circular material element will deform into an ellipse, as indicated in Figs. 2.2.2 and 2.2.4. This can be shown as follows. With respect to the principal axes, an undeformed line element $d\mathbf{X} = dX_1\mathbf{N}_1 + dX_2\mathbf{N}_2$ has magnitude squared $\left(dX_1\right)^2 + \left(dX_2\right)^2 = c^2$, where c is the radius of the circle, Fig. 2.2.9. The deformed element is $d\mathbf{x} = \mathbf{U}d\mathbf{X}$, or $d\mathbf{x} = \lambda_1 dX_1\mathbf{N}_1 + \lambda_2 dX_2\mathbf{N}_2 \equiv dx_1\mathbf{n}_1 + dx_2\mathbf{n}_2$. Thus $dx_1/\lambda_1 = dX_1$, $dx_2/\lambda_2 = dX_2$, which leads to the standard equation of an ellipse with major and minor axes $\lambda_1 c, \lambda_2 c$: $\left(dx_1/\lambda_1 c\right)^2 + \left(dx_2/\lambda_2 c\right)^2 = 1$.

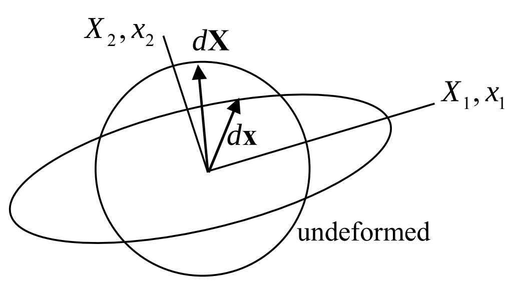


Figure 2.2.9: a circular element deforming into an ellipse

2.2.6 Some Simple Deformations

In this section, some elementary deformations are considered.

Pure Stretch

This deformation has already been seen, but now it can be viewed as a special case of the polar decomposition. The motion is

$$\boxed{x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3} \qquad \textbf{Pure Stretch} \qquad (2.2.38)$$

and the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Here, $\mathbf{R} = \mathbf{I}$ and there is no rotation. $\mathbf{U} = \mathbf{F}$ and the principal material axes are coincident with the material coordinate axes. $\lambda_1, \lambda_2, \lambda_3$, the eigenvalues of **U**, are the principal stretches.

Stretch with rotation

Consider the motion

$$x_1 = X_1 - kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $k = \tan\theta$. This decomposition shows that the deformation consists of material stretching by $\sec\theta (= \sqrt{1+k^2})$, the principal stretches, along each of the axes, followed by a rigid body rotation through an angle θ about the $X_3 = 0$ axis, Fig. 2.2.10. The deformation is relatively simple because the principal material axes are aligned with the material coordinate axes (so that **U** is diagonal). The deformation of the unit square is as shown in Fig. 2.2.10.

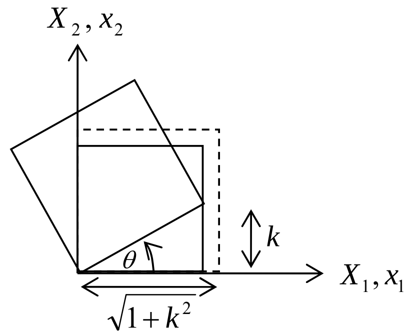


Figure 2.2.10: stretch with rotation

Pure Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3} \qquad \textbf{Pure Shear} \qquad (2.2.39)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where, since **F** is symmetric, there is no rotation, and $\mathbf{F} = \mathbf{U}$. Since the rotation is zero, one can work directly with **U** and not have to consider **C**. The eigenvalues of **U**, the principal stretches, are $1+k, 1-k, 1$, with corresponding principal directions $\hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$, $\hat{\mathbf{N}}_2 = -\frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$ and $\hat{\mathbf{N}}_3 = \mathbf{E}_3$.

The deformation of the unit square is as shown in Fig. 2.2.11. The diagonal indicated by the heavy line stretches by an amount $1+k$ whereas the other diagonal contracts by an amount $1-k$. An element of material along the diagonal will undergo a pure stretch as indicated by the stretching of the dotted box.

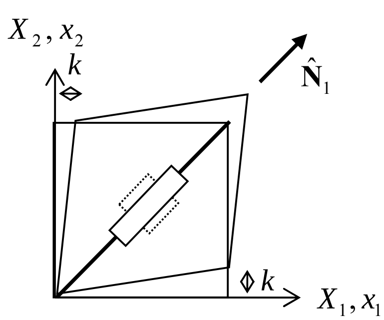


Figure 2.2.11: pure shear

Simple Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3} \qquad \textbf{Simple Shear} \qquad (2.2.40)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The invariants of **C** are $\mathrm{I}_{\mathbf{C}} = 3 + k^2$, $\mathrm{II}_{\mathbf{C}} = 3 + k^2$, $\mathrm{III}_{\mathbf{C}} = 1$ and the characteristic equation is $\lambda^3 + (3 + k^2)\lambda(1-\lambda) - 1 = 0$, so the principal values of **C** are $\lambda = 1 + \frac{1}{2}k^2 \pm \frac{1}{2}k\sqrt{4+k^2}, 1$. The principal values of **U** are the (positive) square-roots of these: $\lambda = \frac{1}{2}\sqrt{4+k^2} \pm \frac{1}{2}k, 1$. These can be written as $\lambda = \sec\theta \pm \tan\theta, 1$ by letting $\tan\theta = \frac{1}{2}k$. The corresponding eigenvectors of **C** are

$$\hat{\mathbf{N}}_1 = \frac{k}{\frac{1}{2}k^2 + \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = \frac{k}{\frac{1}{2}k^2 - \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

or, normalizing so that they are of unit size, and writing in terms of θ,

$$\hat{\mathbf{N}}_1 = \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = -\sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

The transformation matrix of direction cosines is then

$$[\mathbf{Q}] = \begin{bmatrix} \sqrt{(1-\sin\theta)/2} & -\sqrt{(1+\sin\theta)/2} & 0 \\ \sqrt{(1+\sin\theta)/2} & \sqrt{(1-\sin\theta)/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that, using the inverse transformation formula, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$, one obtains **U** in terms of the original coordinates, and hence

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & (1+\sin^2\theta)/\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The deformation of the unit square is shown in Fig. 2.2.12 (for $k = 0.2, \theta = 5.71^{\circ}$). The square first undergoes a pure stretch/contraction ($\hat{\mathbf{N}}_1$ is in this case at 47.86° to the X_1

axis, with the diagonal of the square becoming the diagonal of the parallelogram, at 45.5° to the X_1 axis), and is then brought to its final position by a negative (clockwise) rotation of θ.

For this deformation, $\det \mathbf{F} = 1$ and, as will be shown below, this means that the simple shear deformation is volume-preserving.

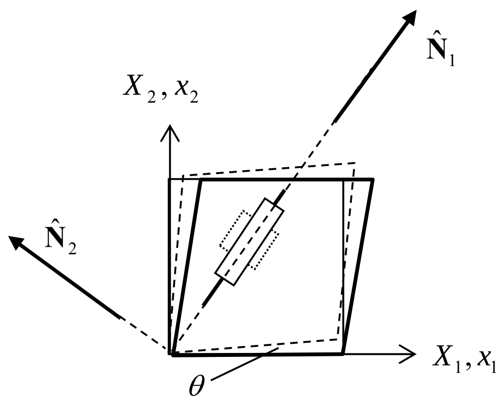


Figure 2.2.12: simple shear

2.2.7 Displacement & Displacement Gradients

The displacement of a material particle[7] is the movement it undergoes in the transition from the reference configuration to the current configuration. Thus, Fig. 2.2.13,[8]

$$\boxed{\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}} \qquad \textbf{Displacement (Material Description)} \tag{2.2.41}$$

$$\boxed{\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t)} \qquad \textbf{Displacement (Spatial Description)} \tag{2.2.42}$$

Note that **U** and **u** have the same values, they just have different arguments.

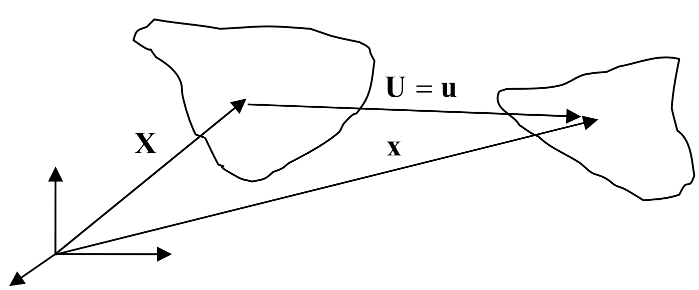


Figure 2.2.13: the displacement

[7] In solid mechanics, the motion and deformation are often described in terms of the displacement **u**. In fluid mechanics, however, the primary field quantity describing the kinematic properties is the velocity **v** (and the acceleration $\mathbf{a} = \dot{\mathbf{v}}$) – see later.
[8] The material displacement **U** here is not to be confused with the right stretch tensor discussed earlier.

Displacement Gradients

The displacement gradient in the material and spatial descriptions, $\partial\mathbf{U}(\mathbf{X},t)/\partial\mathbf{X}$ and $\partial\mathbf{u}(\mathbf{x},t)/\partial\mathbf{x}$, are related to the deformation gradient and the inverse deformation gradient through

$$\begin{aligned}
\mathrm{Grad}\mathbf{U} &= \frac{\partial\mathbf{U}}{\partial\mathbf{X}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial\mathbf{X}} = \mathbf{F}-\mathbf{I} &\qquad \frac{\partial U_i}{\partial X_j} &= \frac{\partial x_i}{\partial X_j} - \delta_{ij} \\
\mathrm{grad}\mathbf{u} &= \frac{\partial\mathbf{u}}{\partial\mathbf{x}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial\mathbf{x}} = \mathbf{I}-\mathbf{F}^{-1} &\qquad \frac{\partial u_i}{\partial x_j} &= \delta_{ij} - \frac{\partial X_i}{\partial x_j}
\end{aligned} \tag{2.2.43}$$

and it is clear that the displacement gradients are related through (see Eqn. 2.2.8)

$$\mathrm{grad}\mathbf{u} = \mathrm{Grad}\,\mathbf{U}\,\mathbf{F}^{-1} \tag{2.2.44}$$

The deformation can now be written in terms of either the material or spatial displacement gradients:

$$\begin{aligned}
d\mathbf{x} &= d\mathbf{X} + d\mathbf{U}(\mathbf{X}) = d\mathbf{X} + \mathrm{Grad}\mathbf{U}\,d\mathbf{X} \\
d\mathbf{x} &= d\mathbf{X} + d\mathbf{u}(\mathbf{x}) = d\mathbf{X} + \mathrm{grad}\mathbf{u}\,d\mathbf{x}
\end{aligned} \tag{2.2.45}$$

Example

Consider again the extension of the bar shown in Fig. 2.1.5. The displacement is

$$\mathbf{U}(\mathbf{X}) = \left(t + 3X_1 t\right)\mathbf{E}_1, \qquad \mathbf{u}(\mathbf{x}) = \left(\frac{t+3x_1 t}{1+3t}\right)\mathbf{e}_1$$

and the displacement gradients are

$$\mathrm{Grad}\mathbf{U} = 3t\mathbf{E}_1, \qquad \mathrm{grad}\mathbf{u} = \left(\frac{3t}{1+3t}\right)\mathbf{e}_1$$

The displacement is plotted in Fig. 2.2.14 for $t=1$. The two gradients $\partial U_1/\partial X_1$ and $\partial u_1/\partial x_1$ have different values (see the horizontal axes on Fig. 2.2.14). In this example, $\partial U_1/\partial X_1 > \partial u_1/\partial x_1$ – the change in displacement is not as large when “seen” from the spatial coordinates.

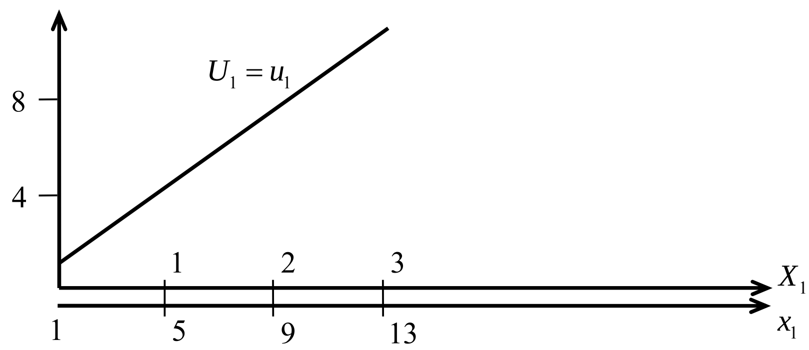


Figure 2.1.14: displacement and displacement gradient

■

Strains in terms of Displacement Gradients

The strains can be written in terms of the displacement gradients. Using 1.10.3b,

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) \\ &= \frac{1}{2}\left(\left(\mathrm{Grad}\mathbf{U} + \mathbf{I}\right)^{\mathrm{T}}\left(\mathrm{Grad}\mathbf{U} + \mathbf{I}\right) - \mathbf{I}\right) \\ &= \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}}\mathrm{Grad}\mathbf{U}\right), \quad E_{IJ} = \frac{1}{2}\left\{\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_K}{\partial X_I}\frac{\partial U_K}{\partial X_J}\right\}\end{aligned} \tag{2.2.46a}$$

$$\begin{aligned}\mathbf{e} &= \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right) \\ &= \frac{1}{2}\left(\mathbf{I} - \left(\mathbf{I} - \mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\left(\mathbf{I} - \mathrm{grad}\mathbf{u}\right)\right) \\ &= \frac{1}{2}\left(\mathrm{grad}\mathbf{u} + \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}} - \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\mathrm{grad}\mathbf{u}\right), \quad e_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\}\end{aligned} \tag{2.2.46b}$$

Small Strain

If the displacement gradients are small, then the quadratic terms, their products, are small relative to the gradients themselves, and may be neglected. With this assumption, the Green-Lagrange strain **E** (and the Euler-Almansi strain) reduces to the **small-strain tensor**,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}}\right), \quad \varepsilon_{IJ} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I}\right) \tag{2.2.47}$$

Since in this case the displacement gradients are small, it does not matter whether one refers the strains to the reference or current configurations – the error is of the same order as the quadratic terms already neglected[9], so the small strain tensor can equally well be written as

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2}\left(\text{grad}\mathbf{u} + (\text{grad}\mathbf{u})^{\mathrm{T}}\right), \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \quad \textbf{Small Strain Tensor} \qquad (2.2.48)$$

2.2.8 The Deformation of Area and Volume Elements

Line elements transform between the reference and current configurations through the deformation gradient. Here, the transformation of area and volume elements is examined.

The Jacobian Determinant

The **Jacobian determinant** of the deformation is defined as the determinant of the deformation gradient,

$$\boxed{J(\mathbf{X},t) = \det\mathbf{F}} \qquad \det\mathbf{F} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \quad \textbf{The Jacobian Determinant} \qquad (2.2.49)$$

Equivalently, it can be considered to be the Jacobian of the transformation from material to spatial coordinates (see Appendix 1.B.2).

From Eqn. 1.3.17, the Jacobian can also be written in the form of the triple scalar product

$$J = \frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3}\right) \qquad (2.2.50)$$

Consider now a volume element in the reference configuration, a parallelepiped bounded by the three line-elements $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$. The volume of the parallelepiped[10] is given by the triple scalar product (Eqns. 1.1.4):

$$dV = d\mathbf{X}^{(1)} \cdot \left(d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)}\right) \qquad (2.2.51)$$

After deformation, the volume element is bounded by the three vectors $d\mathbf{x}^{(i)}$, so that the volume of the deformed element is, using 1.10.16f,

[9] although large rigid body rotations must not be allowed – see §2.7 .
[10] the vectors should form a right-handed set so that the volume is positive.

$$\begin{aligned} dv &= d\mathbf{x}^{(1)} \cdot \left(d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)}\right) \\ &= \mathbf{F}d\mathbf{X}^{(1)} \cdot \left(\mathbf{F}d\mathbf{X}^{(2)} \times \mathbf{F}d\mathbf{X}^{(3)}\right) \\ &= \det \mathbf{F} \left(d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)}\right) \\ &= \det \mathbf{F}\; dV \end{aligned} \qquad (2.2.52)$$

Thus the scalar J is a measure of how the volume of a material element has changed with the deformation and for this reason is often called the **volume ratio**.

$$\boxed{dv = J\, dV} \qquad \textbf{Volume Ratio} \qquad (2.2.53)$$

Since volumes cannot be negative, one must insist on physical grounds that $J > 0$. Also, since **F** has an inverse, $J \neq 0$. Thus one has the restriction

$$J > 0 \qquad (2.2.54)$$

Note that a rigid body rotation does not alter the volume, so the volume change is completely characterised by the stretching tensor **U**. Three line elements lying along the principal directions of **U** form an element with volume dV, and then undergo pure stretch into new line elements defining an element of volume $dv = \lambda_1\lambda_2\lambda_3 dV$, where λ_i are the principal stretches, Fig. 2.2.15. The unit change in volume is therefore also

$$\frac{dv - dV}{dV} = \lambda_1\lambda_2\lambda_3 - 1 \qquad (2.2.55)$$

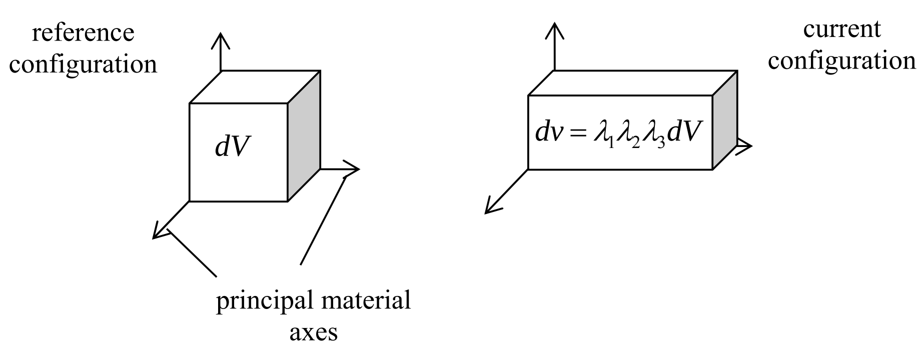


Figure 2.2.15: change in volume

For example, the volume change for pure shear is $-k^2$ (volume decreasing) and, for simple shear, is zero (*cf*. Eqn. 2.2.40 *et seq*., $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)(1) - 1 = 0$).

An **incompressible** material is one for which the volume change is zero, i.e. the deformation is isochoric. For such a material, $J = 1$, and the three principal stretches are not independent, but are constrained by

$$\boxed{\lambda_1\lambda_2\lambda_3 = 1} \qquad \textbf{Incompressibility Constraint} \qquad (2.2.56)$$

Nanson's Formula

Consider an area element in the reference configuration, with area dS, unit normal $\hat{\mathbf{N}}$, and bounded by the vectors $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$, Fig. 2.2.16. Then

$$\hat{\mathbf{N}}dS = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \tag{2.2.57}$$

The volume of the element bounded by the vectors $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and some arbitrary line element $d\mathbf{X}$ is $dV = \hat{\mathbf{N}}dS \cdot d\mathbf{X}$. The area element is now deformed into an element of area ds with normal $\hat{\mathbf{n}}$ and bounded by the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. The volume of the new element bounded by the area element and $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ is then

$$dv = \hat{\mathbf{n}}ds \cdot d\mathbf{x} = \hat{\mathbf{n}}ds \cdot \mathbf{F}d\mathbf{X} \equiv J\hat{\mathbf{N}}dS \cdot d\mathbf{X} \tag{2.2.58}$$

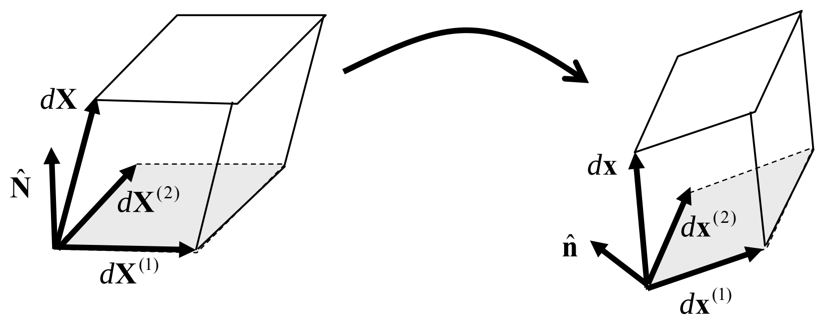


Figure 2.2.16: change of surface area

Thus, since $d\mathbf{X}$ is arbitrary, and using 1.10.3d,

$$\boxed{\hat{\mathbf{n}}ds = J\,\mathbf{F}^{-\mathrm{T}}\hat{\mathbf{N}}dS} \qquad \textbf{Nanson's Formula} \tag{2.2.59}$$

Nanson's formula shows how the vector element of area $\hat{\mathbf{n}}ds$ in the current configuration is related to the vector element of area $\hat{\mathbf{N}}dS$ in the reference configuration.

2.2.9 Inextensibility and Orientation Constraints

A constraint on the principal stretches was introduced for an incompressible material, 2.2.56. Other constraints arise in practice. For example, consider a material which is inextensible in a certain direction, defined by a unit vector $\hat{\mathbf{A}}$ in the reference configuration. It follows that $\left|\mathbf{F}\hat{\mathbf{A}}\right| = 1$ and the constraint can be expressed as 2.2.17,

$$\boxed{\hat{\mathbf{A}}\mathbf{C}\hat{\mathbf{A}} = 1} \qquad \textbf{Inextensibility Constraint} \tag{2.2.60}$$

If there are two such directions in a plane, defined by $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, making angles θ and ϕ respectively with the principal material axes $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$, then

$$1 = \begin{bmatrix}\cos\theta & \sin\theta & 0\end{bmatrix}\begin{bmatrix}\lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix}$$

and $(\lambda_1^2 - \lambda_2^2)\cos^2\theta = 1 - \lambda_2^2 = (\lambda_1^2 - \lambda_2^2)\cos^2\phi$. It follows that $\phi = \theta$, $\phi = \theta + \pi$, $\theta + \phi = \pi$ or $\theta + \phi = 2\pi$ (or $\lambda_1 = \lambda_2 = 1$, i.e. no deformation).

Similarly, one can have orientation constraints. For example, suppose that the direction associated with the vector $\hat{\mathbf{A}}$ maintains that direction. Then

$$\boxed{\mathbf{F}\hat{\mathbf{A}} = \mu\hat{\mathbf{A}}} \qquad \textbf{Orientation Constraint} \qquad (2.2.61)$$

for some scalar $\mu > 0$.

2.2.10 Problems

1. In equations 2.2.8, one has from the chain rule
$$\text{grad}\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \frac{\partial\phi}{\partial X_m}\frac{\partial X_m}{\partial x_i}\mathbf{e}_i = \left(\frac{\partial\phi}{\partial X_j}\mathbf{E}_j\right)\left(\frac{\partial X_m}{\partial x_i}\mathbf{E}_m \otimes \mathbf{e}_i\right) = \text{Grad}\phi\,\mathbf{F}^{-1}$$
Derive the other two relations.
2. Take the dot product $(\lambda d\hat{\mathbf{x}})\cdot(\lambda d\hat{\mathbf{x}})$ in Eqn. 2.2.29. Then use $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$, and 1.10.3e to show that
$$\lambda^2 = \frac{d\mathbf{X}}{|d\mathbf{X}|}\mathbf{U}\cdot\mathbf{U}\frac{d\mathbf{X}}{|d\mathbf{X}|}$$
3. For the deformation
$$x_1 = X_1 + 2X_3, \quad x_2 = X_2 - 2X_3, \quad x_3 = -2X_1 + 2X_2 + X_3$$
   (a) Determine the Deformation Gradient and the Right Cauchy-Green tensors
   (b) Consider the two line elements $d\mathbf{X}^{(1)} = \mathbf{e}_1$, $d\mathbf{X}^{(2)} = \mathbf{e}_2$ (emanating from (0,0,0)). Use the Right Cauchy Green tensor to determine whether these elements in the current configuration ($d\mathbf{x}^{(1)}$, $d\mathbf{x}^{(2)}$) are perpendicular.
   (c) Use the right Cauchy Green tensor to evaluate the stretch of the line element $d\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$, and hence determine whether the element contracts, stretches, or stays the same length after deformation.
   (d) Determine the Green-Lagrange and Eulerian strain tensors
   (e) Decompose the deformation into a stretching and rotation (check that **U** is symmetric and **R** is orthogonal). What are the principal stretches?
4. Derive Equations 2.2.36.
5. For the deformation
$$x_1 = X_1, \quad x_2 = X_2 + X_3, \quad x_3 = aX_2 + X_3$$

(a) Determine the displacement vector in both the material and spatial forms
(b) Determine the displaced location of the particles in the undeformed state which originally comprise
    (i) the plane circular surface $X_1 = 0, \quad X_2^2 + X_3^2 = 1/(1-a^2)$
    (ii) the infinitesimal cube with edges along the coordinate axes of length $dX_i = \varepsilon$

    Sketch the displaced configurations if $a = 1/2$

6. For the deformation

$$x_1 = X_1 + aX_2, \quad x_2 = X_2 + aX_3, \quad x_3 = aX_1 + X_3$$

(a) Determine the displacement vector in both the material and spatial forms
(b) Calculate the full material (Green-Lagrange) strain tensor and the full spatial strain tensor
(c) Calculate the infinitesimal strain tensor as derived from the material and spatial tensors, and compare them for the case of very small *a*.

7. In the example given above on the polar decomposition, §2.2.5, check that the relations $\mathbf{C}\mathbf{n}_i = \lambda\mathbf{n}_i,\ i = 1,2,3$ are satisfied (with respect to the original axes). Check also that the relations $\mathbf{C}\mathbf{n}'_i = \lambda\mathbf{n}'_i,\ i = 1,2,3$ are satisfied (here, the eigenvectors are the unit vectors in the second coordinate system, the principal directions of **C**, and **C** is with respect to these axes, i.e. it is diagonal).

2.3 Deformation and Strain: Further Topics

2.3.1 Volumetric and Isochoric Deformations

When analysing materials which are only slightly incompressible, it is useful to decompose the deformation gradient multiplicatively, according to

$$\mathbf{F} = \left(J^{1/3}\mathbf{I}\right)\overline{\mathbf{F}} = J^{1/3}\overline{\mathbf{F}} \tag{2.3.1}$$

From this definition {▲Problem 1},

$$\det \overline{\mathbf{F}} = 1 \tag{2.3.2}$$

and so $\overline{\mathbf{F}}$ characterises a volume preserving (**distortional** or **isochoric**) deformation. The tensor $J^{1/3}\mathbf{I}$ characterises the volume-changing (**dilational** or **volumetric**) component of the deformation, with $\det\left(J^{1/3}\mathbf{I}\right) = \det \mathbf{F} = J$.

This concept can be carried on to other kinematic tensors. For example, with $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$,

$$\mathbf{C} = J^{2/3}\overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}} \equiv J^{2/3}\overline{\mathbf{C}}. \tag{2.3.3}$$

$\overline{\mathbf{F}}$ and $\overline{\mathbf{C}}$ are called the **modified deformation gradient** and the **modified right Cauchy-Green tensor**, respectively. The square of the stretch is given by

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}} = J^{2/3}\left\{d\hat{\mathbf{X}}\overline{\mathbf{C}}d\hat{\mathbf{X}}\right\} \tag{2.3.4}$$

so that $\lambda = J^{1/3}\overline{\lambda}$, where $\overline{\lambda}$ is the **modified stretch**, due to the action of $\overline{\mathbf{C}}$. Similarly, the **modified principal stretches** are

$$\overline{\lambda}_i = J^{-1/3}\lambda_i, \quad i = 1,2,3 \tag{2.3.5}$$

with

$$\det \overline{\mathbf{F}} = \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3 = 1 \tag{2.3.6}$$

The case of simple shear discussed earlier is an example of an isochoric deformation, in which the deformation gradient and the modified deformation gradient coincide, $J^{1/3}\mathbf{I} = \mathbf{I}$.

2.3.2 Relative Deformation

It is usual to use the configuration at $(\mathbf{X}, t = 0)$ as the reference configuration, and define quantities such as the deformation gradient relative to this reference configuration. As mentioned, any configuration can be taken to be the reference configuration, and a new deformation gradient can be constructed with respect to this new reference configuration. Further, the reference configuration does not have to be fixed, but could be moving also.

In many cases, it is useful to choose the *current* configuration $(\mathbf{x}, t)$ to be the reference configuration, for example when evaluating rates of change of kinematic quantities (see later). To this end, introduce a third configuration: this is the configuration at some time $t = \tau$ and the position of a material particle $\mathbf{X}$ here is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}(\mathbf{X}, \tau)$, where $\boldsymbol{\chi}$ is the motion function. The deformation at this time τ relative to the *current* configuration is called the **relative deformation**, and is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}_{(t)}(\mathbf{x}, \tau)$, as illustrated in Fig. 2.3.1.

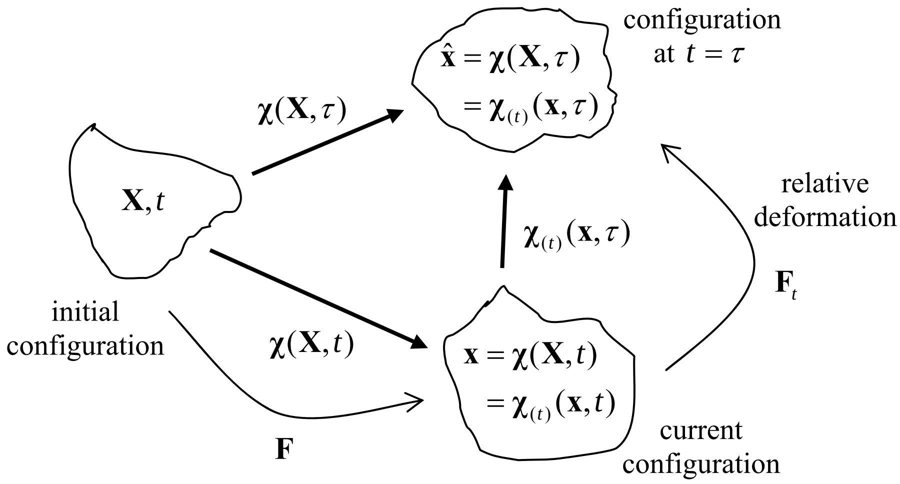


Figure 2.3.1: the relative deformation

The **relative deformation gradient** $\mathbf{F}_t$ is defined through

$$d\hat{\mathbf{x}} = \mathbf{F}_t(\mathbf{x}, \tau)\, d\mathbf{x}, \qquad \mathbf{F}_t = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \tag{2.3.7}$$

Also, since $d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$ and $d\hat{\mathbf{x}} = \mathbf{F}(\mathbf{X}, \tau)\, d\mathbf{X}$, one has the relation

$$\mathbf{F}(\mathbf{X}, \tau) = \mathbf{F}_t(\mathbf{x}, \tau)\, \mathbf{F}(\mathbf{X}, t) \tag{2.3.8}$$

Similarly, relative strain measures can be defined, for example the relative right Cauchy-Green strain tensor is

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^{\mathrm{T}}\, \mathbf{F}_t(\tau) \tag{2.3.9}$$

Example

Consider the two-dimensional motion

$$x_1 = X_1 e^t, \quad x_2 = X_2(t+1)$$

Inverting these gives the spatial description $X_1 = x_1 e^{-t}$, $X_2 = x_2/(t+1)$, and the relative deformation is

$$\hat{x}_1(\mathbf{x},\tau) = X_1 e^{\tau} = x_1 e^{\tau - t}$$
$$\hat{x}_2(\mathbf{x},\tau) = X_2(\tau+1) = x_2(\tau+1)/(t+1)$$

The deformation gradients are

$$\mathbf{F}(\mathbf{X},t) = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j = e^t \mathbf{e}_1 \otimes \mathbf{E}_1 + (t+1)\mathbf{e}_2 \otimes \mathbf{E}_2$$
$$\mathbf{F}_t(\mathbf{x},\tau) = \frac{\partial \hat{x}_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = e^{\tau - t}\mathbf{e}_1 \otimes \mathbf{e}_1 + (\tau+1)/(t+1)\mathbf{e}_2 \otimes \mathbf{e}_2$$

■

2.3.3 Derivatives of the Stretch

In this section, some useful formulae involving the derivatives of the stretches with respect to the Cauchy-Green strain tensors are derived.

Derivatives with respect to b

First, take the stretches to be functions of the left Cauchy-Green strain **b**. Write **b** using the spatial principal directions $\hat{\mathbf{n}}_i$ as a basis, 2.2.37, so that the total differential can be expressed as

$$d\mathbf{b} = \sum_{i=1}^{3} 2\lambda_i d\lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \left[d\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \hat{\mathbf{n}}_i \otimes d\hat{\mathbf{n}}_i \right] \qquad (2.3.10)$$

Since $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = \delta_{ij}$, then

$$\hat{\mathbf{n}}_i d\mathbf{b} \hat{\mathbf{n}}_i = 2\lambda_i d\lambda_i + \lambda_i^2 \left[\hat{\mathbf{n}}_i \cdot d\hat{\mathbf{n}}_i + d\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_i \right] = 2\lambda_i d\lambda_i \quad \text{(no sum over } i\text{)} \qquad (2.3.11)$$

This last follows since the change in a vector of constant length is always orthogonal to the vector itself (as in the curvature analysis of §1.6.2). Using the property $\mathbf{uTv} = \mathbf{T} : (\mathbf{u} \otimes \mathbf{v})$, one has (summing over the *k* but not over the *i*; here $d\lambda_k / d\lambda_i = \delta_{ik}$)

$$d\mathbf{b} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \equiv \frac{\partial \mathbf{b}}{\partial \lambda_k} d\lambda_k : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 2\lambda_i d\lambda_i \quad \rightarrow \quad \frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 1 \qquad (2.3.12)$$

Then, since $\partial \mathbf{b} / \partial \lambda_i : \partial \lambda_i / \partial \mathbf{b}$ is also equal to 1, one has

$$\frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = \frac{\partial \mathbf{b}}{\partial \lambda_i} : \frac{\partial \lambda_i}{\partial \mathbf{b}} \quad \rightarrow \quad \frac{\partial \lambda_i}{\partial \mathbf{b}} = \frac{1}{2\lambda_i}(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \tag{2.3.13}$$

The chain rule then gives the second derivative.

The above analysis is for distinct principal stretches. When $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$, then $\mathbf{b} = \lambda^2 \mathbf{I}$, $d\mathbf{b} = 2\lambda d\lambda \mathbf{I}$. Also, $d\mathbf{b} = 3(\partial \mathbf{b}/\partial \lambda) d\lambda$, so $3(\partial \mathbf{b}/\partial \lambda) = 2\lambda \mathbf{I}$, or

$$3\frac{\partial \mathbf{b}}{\partial \lambda} : \frac{\partial \lambda}{\partial \mathbf{b}} = 2\lambda \mathbf{I} : \frac{\partial \lambda}{\partial \mathbf{b}} \tag{2.3.14}$$

But $\partial \mathbf{b}/\partial \lambda : \partial \lambda / \partial \mathbf{b} = 1$ and $3 = \mathbf{I} : \mathbf{I}$, and so in this case, $\partial \lambda / \partial \mathbf{b} = \mathbf{I}/2\lambda$.

A similar calculation can be carried out for two equal eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. In summary,

$$\begin{array}{ll}
\dfrac{\partial \lambda_i}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_i}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \quad \text{(no sum over } i) & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \\
\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}(\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2) & \\
\dfrac{\partial \lambda_3}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_3}(\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) & \lambda_1 = \lambda_2 = \lambda \neq \lambda_3 \\
\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}\sum_{i=1}^{3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \dfrac{1}{2\lambda}\mathbf{I} & \lambda_1 = \lambda_2 = \lambda_3 = \lambda \\
\dfrac{\partial^2 \lambda_i}{\partial \mathbf{b}^2} = -\dfrac{1}{4\lambda_i^3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \quad \text{(no sum over } i) & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1
\end{array} \tag{2.3.15}$$

Derivatives with respect to C

The stretch can also be considered to be a function of the right Cauchy-Green strain **C**. The derivatives of the stretches with respect to **C** can be found in exactly the same way as for the left Cauchy-Green strain. The results are the same as given in 2.3.15 except that, referring to 2.2.37, **b** is replaced by **C** and $\hat{\mathbf{n}}$ is replaced by $\hat{\mathbf{N}}$.

2.3.4 The Directional Derivative of Kinematic Quantities

The directional derivative of vectors and tensors was introduced in §1.6.11 and §1.15.4. Taking directional derivatives of kinematic quantities is often very useful, for example in linearising equations in order to apply numerical solution algorithms

The Deformation Gradient

First, consider the deformation gradient as a function of the current position **x** (or motion $\boldsymbol{\chi}$) and examine its value at $\mathbf{x} + \mathbf{a}$:

$$\mathbf{F}(\mathbf{x}+\mathbf{a}) = \mathbf{F}(\mathbf{x}) + \partial_{\mathrm{x}}\mathbf{F}[\mathbf{a}] + o(|\mathbf{a}|) \tag{2.3.16}$$

The directional derivative $\partial_{\mathrm{x}}\mathbf{F}[\mathbf{a}] = (\partial\mathbf{F}/\partial\mathbf{x})\mathbf{a}$ can be expressed as

$$\begin{aligned}\partial_{\mathrm{x}}\mathbf{F}[\mathbf{a}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{x}+\varepsilon\mathbf{a}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial(\mathbf{x}+\varepsilon\mathbf{a})}{\partial\mathbf{X}} \\ &= \mathrm{Grad}\,\mathbf{a} \\ &= (\mathrm{grad}\,\mathbf{a})\mathbf{F}\end{aligned} \tag{2.3.17}$$

the last line resulting from 2.2.8b. It follows that the directional derivative of the deformation gradient in the direction of a displacement vector **u** from the *current* configuration is

$$\partial_{\mathrm{x}}\mathbf{F}[\mathbf{u}] = (\mathrm{grad}\,\mathbf{u})\mathbf{F} \tag{2.3.18}$$

On the other hand, consider the deformation gradient as a function of **X** and examine its value at $\mathbf{X}+\mathbf{A}$:

$$\mathbf{F}(\mathbf{X}+\mathbf{A}) = \mathbf{F}(\mathbf{X}) + \partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] \tag{2.3.19}$$

and now

$$\begin{aligned}\partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{X}+\varepsilon\mathbf{A}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}\mathbf{x}(\mathbf{X}+\varepsilon\mathbf{A}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}(\mathbf{x}+\mathbf{F}\varepsilon\mathbf{A}) \\ &= \mathrm{Grad}(\mathbf{F}\mathbf{A}) \\ &= \mathrm{Grad}\,\mathbf{a}\end{aligned} \tag{2.3.20}$$

where $\mathbf{a} = \mathbf{F}\mathbf{A}$.

Other Kinematic Quantities

The directional derivative of the Green-Lagrange strain, the right and left Cauchy-Green tensors and the Jacobian in the direction of a displacement **u** from the current configuration are {▲Problem 2}

$$\begin{aligned}
\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] &= \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}] &= 2\mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}] &= (\mathrm{grad}\mathbf{u})\mathbf{b} + \mathbf{b}(\mathrm{grad}\mathbf{u})^{\mathrm{T}} \\
\partial_{\mathbf{x}}J[\mathbf{u}] &= J\mathrm{div}\mathbf{u}
\end{aligned} \qquad (2.3.21)$$

where $\boldsymbol{\varepsilon}$ is the small-strain tensor, 2.2.48.

The directional derivative is also useful for deriving various relations between the kinematic variables. For example, for an arbitrary vector **a**, using the chain rule 1.15.28, 2.3.20, 1.15.24, the trace relations 1.10.10e and 1.10.10b, and 2.2.8b, 1.14.9,

$$\begin{aligned}
(\mathrm{Grad}J)\cdot\mathbf{a} &= \partial_{\mathbf{X}}J[\mathbf{a}] \\
&= \partial_{\mathbf{F}}J[\partial_{\mathbf{X}}\mathbf{F}[\mathbf{a}]] \\
&= \partial_{\mathbf{F}}J[\mathrm{Grad}(\mathbf{Fa})] \\
&= J\mathbf{F}^{-\mathrm{T}} : \mathrm{Grad}(\mathbf{Fa}) \\
&= J\mathrm{tr}(\mathbf{F}^{-1}\mathrm{Grad}(\mathbf{Fa})) \\
&= J\mathrm{tr}(\mathrm{Grad}(\mathbf{Fa})\mathbf{F}^{-1}) \\
&= J\mathrm{tr}(\mathrm{grad}(\mathbf{Fa})) \\
&= J\,\mathrm{div}(\mathbf{Fa})
\end{aligned} \qquad (2.3.22)$$

so that, from 1.14.16b with **a** constant,

$$\boxed{\mathrm{Grad}J = J\mathrm{div}\mathbf{F}^{\mathrm{T}}} \qquad (2.3.23)$$

2.3.5 Problems

1. Use 1.10.16c to show that $\det\overline{\mathbf{F}} = 1$.
2. (a) use the relation $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$, Eqn. 2.3.18, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = (\mathrm{grad}\mathbf{u})\mathbf{F}$, and the product rule of differentiation to derive 2.3.21a, $\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] = \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F}$, where $\boldsymbol{\varepsilon}$ is the small strain tensor.
   (b) evaluate $\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}]$ (in terms of **F** and $\boldsymbol{\varepsilon}$, the small strain tensor)
   (c) evaluate $\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}]$ (in terms of $\mathrm{grad}\mathbf{u}$ and **b**)
   (d) evaluate $\partial_{\mathbf{x}}J[\mathbf{u}]$ (in terms of *J* and $\mathrm{div}\mathbf{u}$; use the chain rule $\partial_{\mathbf{x}}J[\mathbf{u}] = \partial_{\mathbf{F}}\hat{J}[\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}]]$, with $\hat{J}(\mathbf{F}) = \det\mathbf{F}$, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = \mathrm{Grad}\mathbf{u}$)

2.4 Material Time Derivatives

The motion is now allowed to be a function of time, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, and attention is given to time derivatives, both the **material time derivative** and the **local time derivative**.

2.4.1 Velocity & Acceleration

The velocity of a moving particle is the time rate of change of the position of the particle. From 2.1.3, by definition,

$$\mathbf{V}(\mathbf{X}, t) \equiv \frac{d\boldsymbol{\chi}(\mathbf{X}, t)}{dt} \tag{2.4.1}$$

In the motion expression $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X}$ and t are independent variables and $\mathbf{X}$ is independent of time, denoting the particle for which the velocity is being calculated. The velocity can thus be written as $\partial\boldsymbol{\chi}(\mathbf{X}, t)/\partial t$ or, denoting the motion by $\mathbf{x}(\mathbf{X}, t)$, as $d\mathbf{x}(\mathbf{X}, t)/dt$ or $\partial\mathbf{x}(\mathbf{X}, t)/\partial t$.

The spatial description of the velocity field may be obtained from the material description by simply replacing $\mathbf{X}$ with $\mathbf{x}$, i.e.

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right) \tag{2.4.2}$$

As with displacements in both descriptions, there is only *one* velocity, $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$ – they are just given in terms of different coordinates.

The velocity is most often expressed in the spatial description, as

$$\boxed{\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}} \qquad \textbf{velocity} \tag{2.4.3}$$

To be precise, the right hand side here involves $\mathbf{x}$ which is a function of the material coordinates, but it is understood that the substitution back to spatial coordinates, as in 2.4.2, is made (see example below).

Similarly, the acceleration is defined to be

$$\mathbf{A}(\mathbf{X}, t) = \frac{d^2\boldsymbol{\chi}(\mathbf{X}, t)}{dt^2} = \frac{d^2\mathbf{x}}{dt^2} = \frac{d\mathbf{V}}{dt} = \frac{\partial^2\boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \tag{2.4.4}$$

Example

Consider the motion

$$x_1 = X_1 + t^2 X_2, \quad x_2 = X_2 + t^2 X_1, \quad x_3 = X_3$$

The velocity and acceleration can be evaluated through

$$\mathbf{V}(\mathbf{X},t)=\frac{d\mathbf{x}}{dt}=2tX_2\mathbf{e}_1+2tX_1\mathbf{e}_2,\qquad \mathbf{A}(\mathbf{X},t)=\frac{d^2\mathbf{x}}{dt^2}=2X_2\mathbf{e}_1+2X_1\mathbf{e}_2$$

One can write the motion in the spatial description by inverting the material description:

$$X_1=\frac{x_1-t^2x_2}{1-t^4},\quad X_2=\frac{x_2-t^2x_1}{1-t^4},\quad X_3=x_3$$

Substituting in these equations then gives the spatial description of the velocity and acceleration:

$$\mathbf{v}(\mathbf{x},t)=\mathbf{V}\left(\chi^{-1}(\mathbf{x},t),t\right)=2t\frac{x_2-t^2x_1}{1-t^4}\mathbf{e}_1+2t\frac{x_1-t^2x_2}{1-t^4}\mathbf{e}_2$$
$$\mathbf{a}(\mathbf{x},t)=\mathbf{A}\left(\chi^{-1}(\mathbf{x},t),t\right)=2\frac{x_2-t^2x_1}{1-t^4}\mathbf{e}_1+2\frac{x_1-t^2x_2}{1-t^4}\mathbf{e}_2$$

■

2.4.2 The Material Derivative

One can analyse deformation by examining the current configuration only, discounting the reference configuration. This is the viewpoint taken in Fluid Mechanics – one focuses on material as it flows at the *current time*, and does not consider "where the fluid was". In order to do this, quantities must be cast in terms of the velocity. Suppose that the velocity in terms of spatial coordinates, $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$ is known; for example, one could have a measuring instrument which records the velocity at a specific location, but the motion $\boldsymbol{\chi}$ itself is unknown. In that case, to evaluate the acceleration, the chain rule of differentiation must be applied:

$$\dot{\mathbf{v}}\equiv\frac{d}{dt}\mathbf{v}\left(\mathbf{x}(t),t\right)=\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\frac{d\mathbf{x}}{dt}$$

or

$$\boxed{\mathbf{a}=\frac{\partial\mathbf{v}}{\partial t}+(\mathrm{grad}\,\mathbf{v})\mathbf{v}}\qquad \textbf{acceleration (spatial description)}\qquad (2.4.5)$$

The acceleration can now be determined, because the derivatives can be determined (measured) without knowing the motion.

In the above, the **material derivative**, or **total derivative**, of the particle's velocity was taken to obtain the acceleration. In general, one can take the time derivative of any physical or kinematic property $(\bullet)$ expressed in the spatial description:

$$\boxed{\frac{d}{dt}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \operatorname{grad}(\bullet)\mathbf{v}} \qquad \textbf{Material Time Derivative} \tag{2.4.6}$$

For example, the rate of change of the density $\rho = \rho(\mathbf{x}, t)$ of a particle instantaneously at $\mathbf{x}$ is

$$\dot{\rho} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \operatorname{grad}\rho \cdot \mathbf{v} \tag{2.4.7}$$

The Local Rate of Change

The first term, $\partial\rho / \partial t$, gives the **local rate of change** of density at $\mathbf{x}$ whereas the second term $\mathbf{v} \cdot \operatorname{grad}\rho$ gives the change due to the particle's motion, and is called the **convective rate of change**.

Note the difference between the material derivative and the local derivative. For example, the material derivative of the velocity, 2.4.5 (or, equivalently, $d\mathbf{V}(\mathbf{X}, t)/dt$ in 2.4.4, with $\mathbf{X}$ fixed) is not the same as the derivative $\partial\mathbf{v}(\mathbf{x}, t)/\partial t$ (with $\mathbf{x}$ fixed). The former is the acceleration of a material particle $\mathbf{X}$. The latter is the time rate of change of the velocity of particles *at a fixed location* in space; in general, *different* material particles will occupy position $\mathbf{x}$ at different times.

The material derivative d/dt can be applied to any scalar, vector or tensor:

$$\begin{aligned} \dot{\alpha} &\equiv \frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \operatorname{grad}\alpha \cdot \mathbf{v} \\ \dot{\mathbf{a}} &\equiv \frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + (\operatorname{grad}\mathbf{a})\mathbf{v} \\ \dot{\mathbf{A}} &\equiv \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\operatorname{grad}\mathbf{A})\mathbf{v} \end{aligned} \tag{2.4.8}$$

Another notation often used for the material derivative is D/Dt:

$$\frac{Df}{Dt} \equiv \frac{df}{dt} \equiv \dot{f} \tag{2.4.9}$$

Steady and Uniform Flows

In a **steady flow**, quantities are independent of time, so the local rate of change is zero and, for example, $\dot{\rho} = \operatorname{grad}\rho \cdot \mathbf{v}$. In a **uniform flow**, quantities are independent of position so that, for example, $\dot{\rho} = \partial\rho / \partial t$

Example

Consider again the previous example. This time, with only the velocity $\mathbf{v}(\mathbf{x}, t)$ known, the acceleration can be obtained through the material derivative:

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v}$$

$$= \frac{\partial}{\partial t}\left(2t\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2t\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2\right) + \begin{bmatrix} -\frac{2t^3}{1-t^4} & \frac{2t}{1-t^4} & 0 \\ \frac{2t}{1-t^4} & -\frac{2t^3}{1-t^4} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2t\frac{x_2 - t^2 x_1}{1-t^4} \\ 2t\frac{x_1 - t^2 x_2}{1-t^4} \\ 0 \end{bmatrix}$$

$$= 2\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2$$

as before.

■

The Relationship between the Displacement and Velocity

The velocity can be derived directly from the displacement 2.2.42:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{u}+\mathbf{X})}{dt} = \frac{d\mathbf{u}}{dt}, \tag{2.4.10}$$

or

$$\mathbf{v} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad}\mathbf{u})\mathbf{v} \tag{2.4.11}$$

When the displacement field is given in material form one has

$$\mathbf{V} = \frac{d\mathbf{U}}{dt} \tag{2.4.12}$$

2.4.3 Problems

1. The density of a material is given by

$$\rho = \frac{e^{-2t}}{\mathbf{x}\cdot\mathbf{x}}$$

The velocity field is given by

$$v_1 = x_2 + 2x_3, \quad v_2 = x_3 - 2x_1, \quad v_3 = x_1 + 2x_2$$

Determine the time derivative of the density (a) at a certain position $\mathbf{x}$ in space, and (b) of a material particle instantaneously occupying position $\mathbf{x}$.

2.5 Deformation Rates

In this section, rates of change of the deformation tensors introduced earlier, **F**, **C**, **E**, etc., are evaluated, and special tensors used to measure deformation rates are discussed, for example the velocity gradient **l**, the rate of deformation **d** and the spin tensor **w**.

2.5.1 The Velocity Gradient

The **velocity gradient** is used as a measure of the rate at which a material is deforming.

Consider two fixed neighbouring points, $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, Fig. 2.5.1. The velocities of the material particles at these points at any given time instant are $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x} + d\mathbf{x})$, and

$$\mathbf{v}(\mathbf{x} + d\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x},$$

The relative velocity between the points is

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x} \equiv \mathbf{l} d\mathbf{x} \tag{2.5.1}$$

with **l** defined to be the (spatial) velocity gradient,

$$\boxed{\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \operatorname{grad} \mathbf{v}, \qquad l_{ij} = \frac{\partial v_i}{\partial x_j}} \quad \textbf{Spatial Velocity Gradient} \tag{2.5.2}$$

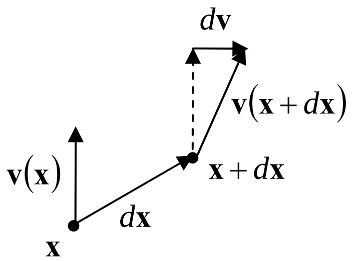


Figure 2.5.1: velocity gradient

Expression 2.5.1 emphasises the tensorial character of the spatial velocity gradient, mapping as it does one vector into another. Its physical meaning will become clear when it is decomposed into its symmetric and skew-symmetric parts below.

The spatial velocity gradient is commonly used in both solid and fluid mechanics. Less commonly used is the material velocity gradient, which is related to the rate of change of the deformation gradient:

$$\operatorname{Grad} \mathbf{V} = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial \mathbf{X}}\right) = \dot{\mathbf{F}} \tag{2.5.3}$$

and use has been made of the fact that, since **X** and *t* are independent variables, material time derivatives and material gradients commute.

2.5.2 Material Derivatives of the Deformation Gradient

The spatial velocity gradient may be written as

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}}{\partial t}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}}$$

or $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ so that the material derivative of **F** can be expressed as

$$\boxed{\dot{\mathbf{F}} = \mathbf{l}\,\mathbf{F}} \qquad \textbf{Material Time Derivative of the Deformation Gradient} \qquad (2.5.4)$$

Also, it can be shown that {▲Problem 1}

$$\boxed{\begin{aligned} \dot{\overline{\mathbf{F}^{\mathrm{T}}}} &= \dot{\mathbf{F}}^{\mathrm{T}} \\ \dot{\overline{\mathbf{F}^{-1}}} &= -\mathbf{F}^{-1}\mathbf{l} \\ \dot{\overline{\mathbf{F}^{-\mathrm{T}}}} &= -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} \end{aligned}} \qquad (2.5.5)$$

2.5.3 The Rate of Deformation and Spin Tensors

The velocity gradient can be decomposed into a symmetric tensor and a skew-symmetric tensor as follows (see §1.10.10):

$$\boxed{\mathbf{l} = \mathbf{d} + \mathbf{w}} \qquad (2.5.6)$$

where **d** is the **rate of deformation tensor** (or **rate of stretching tensor**) and **w** is the **spin tensor** (or **rate of rotation**, or **vorticity tensor**), defined by

$$\boxed{\begin{aligned} \mathbf{d} &= \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ \mathbf{w} &= \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \end{aligned}} \qquad \textbf{Rate of Deformation and Spin Tensors}$$

(2.5.7)

The physical meaning of these tensors is next examined.

The Rate of Deformation

Consider first the rate of deformation tensor **d** and note that

$$\mathbf{l}d\mathbf{x} = d\mathbf{v} = \frac{d}{dt}(d\mathbf{x}) \qquad (2.5.8)$$

The rate at which the square of the length of $d\mathbf{x}$ is changing is then

$$\begin{aligned} \frac{d}{dt}\left(|d\mathbf{x}|^2\right) &= 2|d\mathbf{x}|\frac{d}{dt}\left(|d\mathbf{x}|\right), \\ \frac{d}{dt}\left(|d\mathbf{x}|^2\right) &= \frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = 2d\mathbf{x}\cdot\frac{d}{dt}(d\mathbf{x}) = 2d\mathbf{x}\,\mathbf{l}\,d\mathbf{x} = 2d\mathbf{x}\,\mathbf{d}\,d\mathbf{x} \end{aligned} \qquad (2.5.9)$$

the last equality following from 2.5.6 and 1.10.31e. Dividing across by $2|d\mathbf{x}|^2$, then leads to

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \hat{\mathbf{n}}\,\mathbf{d}\,\hat{\mathbf{n}}} \qquad \textbf{Rate of stretching per unit stretch in the direction } \hat{\mathbf{n}} \qquad (2.5.10)$$

where $\lambda = |d\mathbf{x}|/|d\mathbf{X}|$ is the stretch and $\hat{\mathbf{n}} = d\mathbf{x}/|d\mathbf{x}|$ is a unit normal in the direction of $d\mathbf{x}$. Thus the rate of deformation **d** gives the rate of stretching of line elements. The diagonal components of **d**, for example

$$d_{11} = \mathbf{e}_1\mathbf{d}\mathbf{e}_1,$$

represent unit rates of extension in the coordinate directions.

Note that these are *instantaneous* rates of extension, in other words, they are rates of extensions of elements in the current configuration at the current time; they are not a measure of the rate at which a line element in the original configuration changed into the corresponding line element in the current configuration.

Note:

- Eqn. 2.5.10 can also be derived as follows: let $\hat{\mathbf{N}}$ be a unit normal in the direction of $d\mathbf{X}$, and $\hat{\mathbf{n}}$ be the corresponding unit normal in the direction of $d\mathbf{x}$. Then $\hat{\mathbf{n}}|d\mathbf{x}| = \mathbf{F}\hat{\mathbf{N}}|d\mathbf{X}|$, or $\hat{\mathbf{n}}\lambda = \mathbf{F}\hat{\mathbf{N}}$. Differentiating gives $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \dot{\mathbf{F}}\hat{\mathbf{N}} = \mathbf{l}\mathbf{F}\hat{\mathbf{N}}$ or $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \mathbf{l}\hat{\mathbf{n}}\lambda$. Contracting both sides with $\hat{\mathbf{n}}$ leads to $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} + \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}(\dot{\lambda}/\lambda) = \hat{\mathbf{n}}\mathbf{l}\hat{\mathbf{n}}$. But $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1 \rightarrow d(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})dt = 0$ so, by the chain rule, $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} = 0$ (confirming that a vector $\hat{\mathbf{n}}$ of constant length is orthogonal to a change in that vector $d\hat{\mathbf{n}}$), and the result follows

Consider now the rate of change of the angle θ between two vectors $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. Using 2.5.8 and 1.10.3d,

$$\frac{d}{dt}\left(d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}\right) = \frac{d}{dt}\left(d\mathbf{x}^{(1)}\right) \cdot d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)} \cdot \frac{d}{dt}\left(d\mathbf{x}^{(2)}\right)$$
$$= \mathbf{l}d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)} \cdot \mathbf{l}d\mathbf{x}^{(2)}$$
$$= \left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right)d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}$$
$$= 2\, d\mathbf{x}^{(1)}\mathbf{d}d\mathbf{x}^{(2)} \qquad (2.5.11)$$

which reduces to 2.5.9 when $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)}$. An alternative expression for this dot product is

$$\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\cos\theta\right) = \frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)\left|d\mathbf{x}^{(2)}\right|\cos\theta + \frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)\left|d\mathbf{x}^{(1)}\right|\cos\theta - \sin\theta\,\dot{\theta}\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|$$
$$= \left(\frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)}{\left|d\mathbf{x}^{(1)}\right|}\cos\theta + \frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)}{\left|d\mathbf{x}^{(2)}\right|}\cos\theta - \sin\theta\,\dot{\theta}\right)\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|$$
$$(2.5.12)$$

Equating 2.5.11 and 2.5.12 leads to

$$2\,\hat{\mathbf{n}}_1\,\mathbf{d}\,\hat{\mathbf{n}}_2 = \left(\frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2}\right)\cos\theta - \sin\theta\,\dot{\theta} \qquad (2.5.13)$$

where $\lambda_i = \left|d\mathbf{x}^{(i)}\right| / \left|d\mathbf{X}^{(i)}\right|$ is the stretch and $\hat{\mathbf{n}}_i = d\mathbf{x}^{(i)} / \left|d\mathbf{x}^{(i)}\right|$ is a unit normal in the direction of $d\mathbf{x}^{(i)}$.

It follows from 2.5.13 that the off-diagonal terms of the rate of deformation tensor represent **shear rates**: the rate of change of the right angle between line elements aligned with the coordinate directions. For example, taking the base vectors $\mathbf{e}_1 = \hat{\mathbf{n}}_1$, $\mathbf{e}_2 = \hat{\mathbf{n}}_2$, 2.5.13 reduces to

$$d_{12} = -\frac{1}{2}\dot{\theta}_{12} \qquad (2.5.14)$$

where θ_{12} is the instantaneous right angle between the axes in the current configuration.

The Spin

Consider now the spin tensor $\mathbf{w}$; since it is skew-symmetric, it can be written in terms of its axial vector $\boldsymbol{\omega}$ (Eqn. 1.10.34), called the **angular velocity vector**:

$$\begin{aligned}\boldsymbol{\omega} &= -w_{23}\mathbf{e}_1 + w_{13}\mathbf{e}_2 - w_{12}\mathbf{e}_3 \\ &= \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \\ &= \frac{1}{2}\operatorname{curl}\mathbf{v}\end{aligned} \tag{2.5.15}$$

(The vector $2\boldsymbol{\omega}$ is called the **vorticity** (or **spin**) **vector**.) Thus when **d** is zero, the motion consists of a rotation about some axis at angular velocity $\omega = |\boldsymbol{\omega}|$ (*cf*. the end of §1.10.11), with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\mathbf{r}$ measured from a point on the axis, and $\mathbf{wr} = \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}$.

On the other hand, when $\mathbf{l} = \mathbf{d}$, $\mathbf{w} = \mathbf{0}$, one has $\boldsymbol{\omega} = \mathbf{o}$, and the motion is called **irrotational**.

Example (Shear Flow)

Consider a **simple shear flow** in which the velocity profile is "triangular" as shown in Fig. 2.5.2. This type of flow can be generated (at least approximately) in many fluids by confining the fluid between plates a distance *h* apart, and by sliding the upper plate over the lower one at constant velocity *V*. If the material particles adjacent to the upper plate have velocity $V\mathbf{e}_1$, then the velocity field is $\mathbf{v} = \dot{\gamma}x_2\mathbf{e}_1$, where $\dot{\gamma} = V/h$. This is a steady flow ($\partial\mathbf{v}/\partial t = \mathbf{0}$); at any given point, there is no change over time. The velocity gradient is $\mathbf{l} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2$ and the acceleration of material particles is zero: $\mathbf{a} = \mathbf{lv} = \mathbf{0}$. The rate of deformation and spin are

$$\mathbf{d} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{w} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and, from 2.5.14, $\dot{\gamma} = -\dot{\theta}_{12}$, the rate of change of the angle shown in Fig. 2.5.2.

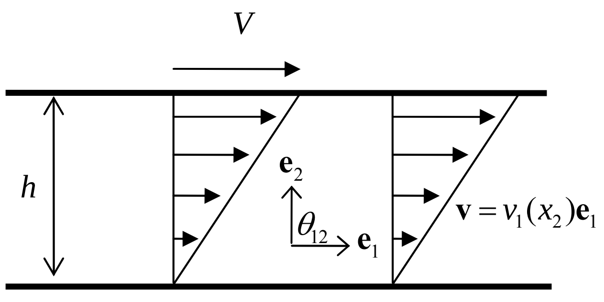


Figure 2.5.2: shear flow

The eigenvalues of **d** are $\lambda = 0, \pm\dot{\gamma}/2$ ($\det\mathbf{d} = 0$) and the principal invariants, Eqn. 1.11.17, are $\mathrm{I}_\mathbf{d} = 0, \mathrm{II}_\mathbf{d} = -\frac{1}{4}\dot{\gamma}^2, \mathrm{III}_\mathbf{d} = 0$. For $\lambda = +\dot{\gamma}/2$, the eigenvector is $\mathbf{n}_1 = [1 \quad 1 \quad 0]^T$ and for $\lambda = -\dot{\gamma}/2$, it is $\mathbf{n}_2 = [-1 \quad 1 \quad 0]^T$ (for $\lambda = 0$ it is $\mathbf{e}_3$). (The eigenvalues and eigenvectors of **w** are complex.) Relative to the basis of eigenvectors,

$$\mathbf{d} = \begin{bmatrix} \dot{\gamma}/2 & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so at 45° there is an instantaneous pure rate of stretching/contraction of material.

■

2.5.4 Other Rates of Strain Tensors

From 2.2.9, 2.2.22,

$$\frac{1}{2}\frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = d\mathbf{X}\frac{1}{2}\dot{\mathbf{C}}d\mathbf{X} = d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} \tag{2.5.16}$$

This can also be written in terms of spatial line elements:

$$d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} = d\mathbf{x}\left[\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1}\right]d\mathbf{x} \tag{2.5.17}$$

But from 2.5.9, these also equal $d\mathbf{x}\mathbf{d}d\mathbf{x}$, which leads to expressions for the material time derivatives of the right Cauchy-Green and Green-Lagrange strain tensors (also given here are expressions for the time derivatives of the left Cauchy-Green and Euler-Almansi tensors {▲Problem 3})

$$\boxed{\begin{aligned} \dot{\mathbf{C}} &= 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ \dot{\mathbf{E}} &= \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ \dot{\mathbf{b}} &= \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}} \\ \dot{\mathbf{e}} &= \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l} \end{aligned}} \tag{2.5.18}$$

Note that

$$\int \dot{\mathbf{E}}\,dt = \int d\mathbf{E}$$

so that the integral of the rate of Green-Lagrange strain is path independent and, in particular, the integral of $\dot{\mathbf{E}}$ around any closed loop (so that the final configuration is the same as the initial configuration) is zero. However, in general, the integral of the rate of deformation,

$$\int \mathbf{d}dt$$

is not independent of the path – there is no universal function $\mathbf{h}$ such that $\mathbf{d} = d\mathbf{h}/dt$ with $\int \mathbf{d}dt = \int d\mathbf{h}$. Thus the integral $\int \mathbf{d}dt$ over a closed path may be non-zero, and hence the integral of the rate of deformation is not a good measure of the total strain.

The Hencky Strain

The Hencky strain is, Eqn. 2.2.37, $\mathbf{h} = \sum_{i=1}^{3} (\ln \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, where $\mathbf{n}_i$ are the principal spatial axes. Thus, if the principal spatial axes do not change with time, $\dot{\mathbf{h}} = \sum_{i=1}^{3} (\dot{\lambda}_i / \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$. With the left stretch $\mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, it follows that (and similarly for the corresponding material tensors), $\dot{\mathbf{H}} \equiv \dot{\overline{\ln \mathbf{U}}} = \dot{\mathbf{U}}\mathbf{U}^{-1}$, $\dot{\mathbf{h}} \equiv \dot{\overline{\ln \mathbf{v}}} = \dot{\mathbf{v}}\mathbf{v}^{-1}$.

For example, consider an extension in the coordinate directions, so $\mathbf{F} = \mathbf{U} = \mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i$. The motion and velocity are

$$x_i = \lambda_i X_i, \quad \dot{x}_i = \dot{\lambda}_i X_i = \frac{\dot{\lambda}_i}{\lambda_i} x_i \qquad (\text{no sum})$$

so $d_i = \dot{\lambda}_i / \lambda_i$ (no sum), and $\mathbf{d} = \dot{\mathbf{h}}$. Further, $\mathbf{h} = \int \mathbf{d} dt$. Note that, as mentioned above, this expression does not hold in general, but does in this case of uniform extension.

2.5.5 Material Derivatives of Line, Area and Volume Elements

The material derivative of a line element $d(d\mathbf{x})/dt$ has been derived (defined) through 2.4.8. For area and volume elements, it is necessary first to evaluate the material derivative of the Jacobian determinant *J*. From the chain rule, one has (see Eqns 1.15.11, 1.15.7)

$$\dot{J} = \frac{d}{dt}\left(J(\mathbf{F})\right) = \frac{\partial J}{\partial \mathbf{F}} : \dot{\mathbf{F}} = J\mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} \tag{2.5.19}$$

Hence {▲Problem 4}

$$\boxed{\begin{aligned} \dot{J} &= J\,\mathrm{tr}(\mathbf{l}) \\ &= J\,\mathrm{tr}(\mathrm{grad}\,\mathbf{v}) \\ &= J\mathrm{div}\mathbf{v} \end{aligned}} \tag{2.5.20}$$

Since $\mathbf{l} = \mathbf{d} + \mathbf{w}$ and $\mathrm{tr}\mathbf{w} = 0$, it also follows that $\dot{J} = J\mathrm{tr}\mathbf{d}$.

As mentioned earlier, an isochoric motion is one for which the volume is constant – thus any of the following statements characterise the necessary and sufficient conditions for an isochoric motion:

$$J = 1, \quad \dot{J} = 0, \quad \mathrm{div}\mathbf{v} = 0, \quad \mathrm{tr}\,\mathbf{d} = 0, \quad \mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} = 0 \tag{2.5.21}$$

Applying Nanson's formula 2.2.59, the material derivative of an area vector element is {▲Problem 6}

$$\boxed{\frac{d}{dt}(\hat{\mathbf{n}}ds) = \left(\operatorname{div}\mathbf{v} - \mathbf{l}^{\mathrm{T}}\right)\hat{\mathbf{n}}ds} \tag{2.5.22}$$

Finally, from 2.2.53, the material time derivative of a volume element is

$$\boxed{\frac{d}{dt}(dv) = \frac{d}{dt}(JdV) = \dot{J}dV = \operatorname{div}\mathbf{v}\,dv} \tag{2.5.23}$$

Example (Shear and Stretch)

Consider a sample of material undergoing the following motion, Fig. 2.4.3.

$$\begin{aligned} x_1 &= X_1 + k\lambda X_2 \\ x_2 &= \lambda X_2 \\ x_3 &= X_3 \end{aligned}, \qquad \begin{aligned} X_1 &= x_1 - kx_2 \\ X_2 &= \frac{1}{\lambda}x_2 \\ X_3 &= x_3 \end{aligned}$$

with $\lambda = \lambda(t)$, $k = k(t)$.

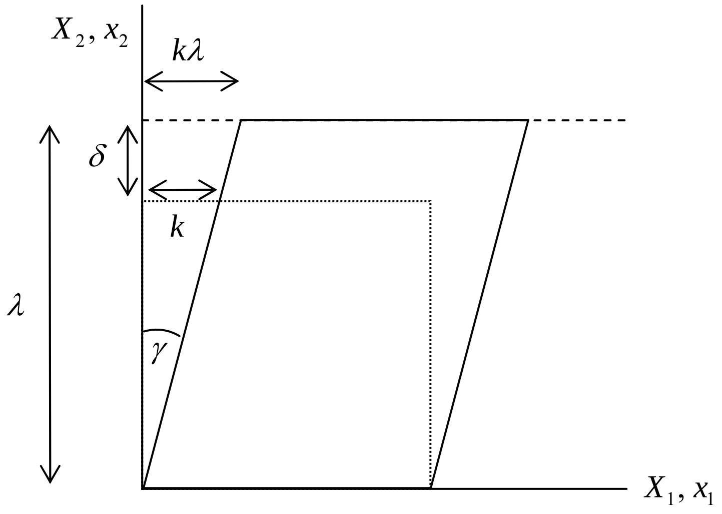


Figure 2.4.3: shear and stretch

The deformation gradient and material strain tensors are

$$\mathbf{F} = \begin{bmatrix} 1 & k\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & k\lambda & 0 \\ k\lambda & (1+k^2)\lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & \frac{1}{2}k\lambda & 0 \\ \frac{1}{2}k\lambda & \frac{1}{2}\left(\lambda^2(1+k^2)-1\right) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the Jacobian $J = \det\mathbf{F} = \lambda$, and the spatial strain tensors are

$$\mathbf{b}=\begin{bmatrix}1+k^2\lambda^2 & k\lambda^2 & 0\\ k\lambda^2 & \lambda^2 & 0\\ 0 & 0 & 1\end{bmatrix},\qquad \mathbf{e}=\begin{bmatrix}0 & \frac{1}{2}k & 0\\ \frac{1}{2}k & \frac{1}{2}\dfrac{\left(1-k^2\right)\lambda^2-1}{\lambda^2} & 0\\ 0 & 0 & 0\end{bmatrix}$$

This deformation can also be expressed as a stretch followed by a simple shear:

$$\mathbf{F}=\begin{bmatrix}1 & k & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & 1\end{bmatrix}$$

The velocity is

$$\mathbf{V}=\frac{d\mathbf{x}}{dt}=\begin{bmatrix}\left(\dot{k}\lambda+k\dot{\lambda}\right)X_2\\ \dot{\lambda}X_2\\ 0\end{bmatrix},\qquad \mathbf{v}=\begin{bmatrix}\left(\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right)x_2\\ \left(\dot{\lambda}/\lambda\right)x_2\\ 0\end{bmatrix}$$

The velocity gradient is

$$\mathbf{l}=\frac{d\mathbf{v}}{d\mathbf{x}}=\begin{bmatrix}0 & \dot{k}+k\left(\dot{\lambda}/\lambda\right) & 0\\ 0 & \dot{\lambda}/\lambda & 0\\ 0 & 0 & 0\end{bmatrix}$$

and the rate of deformation and spin are

$$\mathbf{d}=\begin{bmatrix}0 & \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0\\ \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & \dot{\lambda}/\lambda & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}0 & \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0\\ -\frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Also

$$\dot{\mathbf{C}}=2\mathbf{F}^T\mathbf{d}\mathbf{F}=\begin{bmatrix}0 & \lambda\dot{k}+k\dot{\lambda} & 0\\ \lambda\dot{k}+k\dot{\lambda} & 2\lambda\left(k\lambda\dot{k}+\left(k^2+1\right)\dot{\lambda}\right) & 0\\ 0 & 0 & 0\end{bmatrix}$$

As expected, from 2.5.20,

$$\dot{J}=J\mathrm{tr}(\mathbf{d})=J\left(\dot{\lambda}/\lambda\right)=\dot{\lambda}$$

■

2.5.6 Problems

1. (a) Differentiate the relation $\mathbf{I} = \mathbf{F}\mathbf{F}^{-1}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, to derive 2.5.5b,

   $\overline{\mathbf{F}^{-1}}^{\,\cdot} = -\mathbf{F}^{-1}\mathbf{l}$.

   (b) Differentiate the relation $\mathbf{I} = \mathbf{F}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, and 1.10.3e to derive 2.5.5c, $\overline{\mathbf{F}^{-\mathrm{T}}}^{\,\cdot} = -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$.

2. For the velocity field

   $$v_1 = x_1^2 x_2, \quad v_2 = 2x_2^2 x_3, \quad v_3 = 3x_1 x_2 x_3$$

   determine the rate of stretching per unit stretch at (2,0,1) in the direction of the unit vector

   $$(4\mathbf{e}_1 - 3\mathbf{e}_2)/5$$

   And in the direction of $\mathbf{e}_1$?

3. (a) Derive the relation 2.5.18a, $\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$ directly from $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$

   (b) Use the definitions $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{e} = (\mathbf{I} - \mathbf{b}^{-1})/2$ to derive the relations 2.5.18c,d: $\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}}$, $\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l}$

4. Use 2.5.4, 2.5.19, 1.10.3h, 1.10.6, to derive 2.5.20.

5. For the motion $x_1 = 3X_1 t - t^2,\ x_2 = X_1 + X_2 t,\ x_3 = tX_3$, verify that $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$. What is the ratio of the volume element currently occupying (1,1,1) to its volume in the undeformed configuration? And what is the rate of change of this volume element, per unit current volume?

6. Use Nanson's formula 2.2.59, the product rule of differentiation, and 2.5.20, 2.5.5c, to derive the material time derivative of a vector area element, 2.5.22 (note that $\hat{\mathbf{N}}$, a unit normal in the undeformed configuration, is constant).

2.6 Deformation Rates: Further Topics

2.6.1 Relationship between l, d, w and the rate of change of R and U

Consider the polar decomposition $\mathbf{F} = \mathbf{RU}$. Since $\mathbf{R}$ is orthogonal, $\mathbf{RR}^{\mathrm{T}} = \mathbf{I}$, and a differentiation of this equation leads to

$$\mathbf{\Omega_R} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \tag{2.6.1}$$

with $\mathbf{\Omega_R}$ skew-symmetric (see Eqn. 1.14.2). Using this relation, the expression $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and the definitions of **d** and **w**, Eqn. 2.5.7, one finds that {▲Problem 1}

$$\begin{aligned}
\mathbf{l} &= \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} + \mathbf{\Omega_R} \\
\mathbf{w} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} + \mathbf{\Omega_R} \\
&= \mathbf{R}\mathrm{skew}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}} + \mathbf{\Omega_R} \\
\mathbf{d} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} \\
&= \mathbf{R}\mathrm{sym}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}}
\end{aligned} \tag{2.6.2}$$

Note that $\mathbf{\Omega_R}$ being skew-symmetric is consistent with **w** being skew-symmetric, and that both **w** and **d** involve **R**, and the rate of change of **U**.

When the motion is a rigid body rotation, then $\dot{\mathbf{U}} = \mathbf{0}$, and

$$\mathbf{w} = \mathbf{\Omega_R} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{2.6.3}$$

2.6.2 Deformation Rate Tensors and the Principal Material and Spatial Bases

The rate of change of the stretch tensor in terms of the principal material base vectors is

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\left\{\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \lambda_i\dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i\right\} \tag{2.6.4}$$

Consider the case when the principal material axes stay constant, as can happen in some simple deformations. In that case, $\dot{\mathbf{U}}$ and $\mathbf{U}^{-1}$ are coaxial (see §1.11.5):

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \quad \text{and} \quad \mathbf{U}^{-1} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \tag{2.6.5}$$

with $\dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}}$ and, as expected, from 2.5.25b, $\mathbf{w} = \mathbf{\Omega_R} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, that is, any spin is due to rigid body rotation.

Similarly, from 2.2.37, and differentiating $\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \mathbf{I}$,

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \left\{ \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i \right\}. \tag{2.6.6}$$

Also, differentiating $\hat{\mathbf{N}}_i \cdot \hat{\mathbf{N}}_j = \delta_{ij}$ leads to $\dot{\hat{\mathbf{N}}}_i \cdot \hat{\mathbf{N}}_j = -\hat{\mathbf{N}}_i \cdot \dot{\hat{\mathbf{N}}}_j$ and so the expression

$$\dot{\hat{\mathbf{N}}}_i = \sum_{m=1}^{3} W_{im} \hat{\mathbf{N}}_m \tag{2.6.7}$$

is valid provided W_{ij} are the components of a skew-symmetric tensor, $W_{ij} = -W_{ji}$. This leads to an alternative expression for the Green-Lagrange tensor:

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \sum_{\substack{m,n=1 \\ m \neq n}}^{3} \tfrac{1}{2} W_{mn} \left(\lambda_m^2 - \lambda_n^2\right) \hat{\mathbf{N}}_m \otimes \hat{\mathbf{N}}_n \tag{2.6.8}$$

Similarly, from 2.2.37, the left Cauchy-Green tensor can be expressed in terms of the principal spatial base vectors:

$$\mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \quad \dot{\mathbf{b}} = \sum_{i=1}^{3} \left\{ 2\lambda_i \dot{\lambda}_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.9}$$

Then, from inspection of 2.5.18c, $\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}}$, the velocity gradient can be expressed as {▲Problem 2}

$$\mathbf{l} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i \right\} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.7}$$

2.6.3 Rates of Change and the Relative Deformation

Just as the material time derivative of the deformation gradient is defined as

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}\mathbf{F}(\mathbf{X}, t) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)$$

one can define the material time derivative of the relative deformation gradient, *cf.* §2.3.2, the rate of change *relative to the current configuration*:

$$\dot{\mathbf{F}}_t(\mathbf{x}, t) = \frac{\partial}{\partial \tau}\mathbf{F}_t(\mathbf{x}, \tau)\Big|_{\tau = t} \tag{2.6.8}$$

From 2.3.8, $\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{F}(\mathbf{X},\tau)\mathbf{F}(\mathbf{X},t)^{-1}$, so taking the derivative with respect to τ (t is now fixed) and setting $\tau = t$ gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \dot{\mathbf{F}}(\mathbf{X},t)\mathbf{F}(\mathbf{X},t)^{-1}$$

Then, from 2.5.4,

$$\mathbf{l} = \dot{\mathbf{F}}_t(\mathbf{x},t) \tag{2.6.9}$$

as expected – the velocity gradient is the rate of change of deformation relative to the current configuration. Further, using the polar decomposition,

$$\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{R}_t(\mathbf{x},\tau)\mathbf{U}_t(\mathbf{x},\tau)$$

Differentiating with respect to τ and setting $\tau = t$ then gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \mathbf{R}_t(\mathbf{x},t)\dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t)\mathbf{U}_t(\mathbf{x},t)$$

Relative to the current configuration, $\mathbf{R}_t(\mathbf{x},t) = \mathbf{U}_t(\mathbf{x},t) = \mathbf{I}$, so, from 2.4.34,

$$\mathbf{l} = \dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t) \tag{2.6.10}$$

With **U** symmetric and **R** skew-symmetric, $\dot{\mathbf{U}}_t(\mathbf{x},t)$, $\dot{\mathbf{R}}_t(\mathbf{x},t)$ are, respectively, symmetric and skew-symmetric, and it follows that

$$\begin{aligned} \mathbf{d} &= \dot{\mathbf{U}}_t(\mathbf{x},t) \\ \mathbf{w} &= \dot{\mathbf{R}}_t(\mathbf{x},t) \end{aligned} \tag{2.6.11}$$

again, as expected – the rate of deformation is the instantaneous rate of stretching and the spin is the instantaneous rate of rotation.

The Corotational Derivative

The **corotational derivative** of a vector **a** is $\overset{\circ}{\mathbf{a}} \equiv \dot{\mathbf{a}} - \mathbf{w}\mathbf{a}$. Formally, it is defined through

$$\begin{aligned} \overset{\circ}{\mathbf{a}} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{R}_t(t+\Delta t)\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{R}_t(t) + \Delta t\dot{\mathbf{R}}_t(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{I} + \Delta t\mathbf{w}(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{a}(t)\right\} - \mathbf{w}(t)\mathbf{a}(t) \\ &= \dot{\mathbf{a}} - \mathbf{w}\mathbf{a} \end{aligned} \tag{2.6.12}$$

The definition shows that the corotational derivative involves taking a vector **a** in the current configuration and rotating it with the rigid body rotation part of the motion, Fig. 2.6.1. It is this new, rotated, vector which is compared with the vector $\mathbf{a}(t+\Delta t)$, which has undergone rotation and stretch.

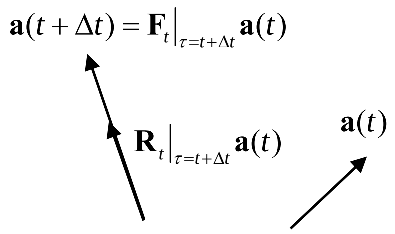


Figure 2.6.1: rotation and stretch of a vector

2.6.4 Rivlin-Ericksen Tensors

The *n*-th **Rivlin-Ericksen tensor** is defined as

$$\mathbf{A}_n(t)=\left.\frac{d^n}{d\tau^n}\mathbf{C}_t(\tau)\right|_{\tau=t}, \qquad n=0,1,2,\cdots \tag{2.6.13}$$

where $\mathbf{C}_t(\tau)$ is the relative right Cauchy-Green strain. Since $\mathbf{C}_t(\tau)\big|_{\tau=t}=\mathbf{I}$, $\mathbf{A}_0=\mathbf{I}$. To evaluate the next Rivlin-Ericksen tensor, one needs the derivatives of the relative deformation gradient; from 2.5.4, 2.3.8,

$$\frac{d}{d\tau}\mathbf{F}_t(\tau)=\frac{d}{d\tau}\left[\mathbf{F}(\tau)\mathbf{F}(t)^{-1}\right]=\mathbf{l}(\tau)\mathbf{F}(\tau)\mathbf{F}(t)^{-1}=\mathbf{l}(\tau)\mathbf{F}_t(\tau) \tag{2.6.14}$$

Then, with 2.5.5a, $d\left(\mathbf{F}_t(\tau)^{\mathrm{T}}\right)/d\tau=\mathbf{F}_t(\tau)^{\mathrm{T}}\mathbf{l}(\tau)^{\mathrm{T}}$, and

$$\begin{aligned}\mathbf{A}_1(t)&=\left[\mathbf{F}_t(\tau)^{\mathrm{T}}\left(\mathbf{l}(\tau)+\mathbf{l}(\tau)^{\mathrm{T}}\right)\mathbf{F}_t(\tau)\right]_{\tau=t}\\&=\left(\mathbf{l}(t)+\mathbf{l}(t)^{\mathrm{T}}\right)\\&=2\mathbf{d}\end{aligned}$$

Thus the tensor $\mathbf{A}_1$ gives a measure of the rate of stretching of material line elements (see Eqn. 2.5.10). Similarly, higher Rivlin-Ericksen tensors give a measure of higher order stretch rates, $\ddot{\lambda}, \dddot{\lambda}$, and so on.

2.6.5 The Directional Derivative and the Material Time Derivative

The directional derivative of a function $\mathbf{T}(t)$ in the direction of an increment in t is, by definition (see, for example, Eqn. 1.15.27),

$$\partial_t \mathbf{T}[\Delta t] = \mathbf{T}(t + \Delta t) - \mathbf{T}(t) \tag{2.6.15}$$

or

$$\partial_t \mathbf{T}[\Delta t] = \frac{d\mathbf{T}}{dt} \Delta t \tag{2.6.16}$$

Setting $\Delta t = 1$, and using the chain rule 1.15.28,

$$\begin{aligned} \dot{\mathbf{T}} &= \partial_t \mathbf{T}[1] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\partial_t \mathbf{x}[1]\right] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\mathbf{v}\right] \end{aligned} \tag{2.6.17}$$

The material time derivative is thus equivalent to the directional derivative in the direction of the velocity vector.

2.6.6 Problems

1. Derive the relations 2.6.2.
2. Use 2.6.9 to verify 2.5.18, $\dot{\mathbf{b}} = \mathbf{lb} + \mathbf{bl}^{\mathrm{T}}$.

2.7 Small Strain Theory

When the deformation is small, from 2.2.43-4,

$$\begin{aligned}\mathbf{F} &= \mathbf{I} + \mathrm{Grad}\mathbf{U} \\ &= \mathbf{I} + (\mathrm{grad}\mathbf{u})\mathbf{F} \\ &\approx \mathbf{I} + \mathrm{grad}\mathbf{u}\end{aligned} \tag{2.7.1}$$

neglecting the product of $\mathrm{grad}\mathbf{u}$ with $\mathrm{Grad}\mathbf{U}$, since these are small quantities. Thus one can take $\mathrm{Grad}\mathbf{U} = \mathrm{grad}\mathbf{u}$ and there is no distinction to be made between the undeformed and deformed configurations. The deformation gradient is of the form $\mathbf{F} = \mathbf{I} + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is small.

2.7.1 Decomposition of Strain

Any second order tensor can be decomposed into its symmetric and antisymmetric part according to 1.10.28, so that

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) + \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) = \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \\ \frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij} + \Omega_{ij}\end{aligned} \tag{2.7.2}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor 2.2.48 and $\boldsymbol{\Omega}$, the anti-symmetric part of the displacement gradient, is the **small rotation tensor**, so that **F** can be written as

$$\boxed{\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}} \quad \textbf{Small Strain Decomposition of the Deformation Gradient} \tag{2.7.3}$$

It follows that (for the calculation of **e**, one can use the relation $(\mathbf{I} + \boldsymbol{\delta})^{-1} \approx \mathbf{I} - \boldsymbol{\delta}$ for small $\boldsymbol{\delta}$)

$$\begin{aligned}\mathbf{C} &= \mathbf{b} = \mathbf{I} + 2\boldsymbol{\varepsilon} \\ \mathbf{E} &= \mathbf{e} = \boldsymbol{\varepsilon}\end{aligned} \tag{2.7.4}$$

Rotation

Since $\boldsymbol{\Omega}$ is antisymmetric, it can be written in terms of an axial vector $\boldsymbol{\omega}$, *cf.* §1.10.11, so that for any vector **a**,

$$\boldsymbol{\Omega}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}, \qquad \boldsymbol{\omega} = -\Omega_{23}\mathbf{e}_1 + \Omega_{13}\mathbf{e}_1 - \Omega_{12}\mathbf{e}_3 \tag{2.7.5}$$

The relative displacement can now be written as

$$\begin{aligned} d\mathbf{u} &= (\operatorname{grad}\mathbf{u})d\mathbf{X} \\ &= \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X} \end{aligned} \qquad (2.7.6)$$

The component of relative displacement given by $\boldsymbol{\omega} \times d\mathbf{X}$ is perpendicular to $d\mathbf{X}$, and so represents a pure rotation of the material line element, Fig. 2.7.1.

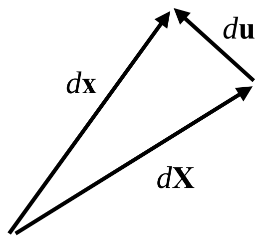


Figure 2.7.1: a pure rotation

Principal Strains

Since $\boldsymbol{\varepsilon}$ is symmetric, it must have three mutually orthogonal eigenvectors, the **principal axes of strain**, and three corresponding real eigenvalues, the **principal strains,** e_1, e_2, e_3), which can be positive or negative, *cf.* §1.11. The effect of $\boldsymbol{\varepsilon}$ is therefore to deform an elemental unit sphere into an elemental ellipsoid, whose axes are the principal axes, and whose lengths are $1+e_1, 1+e_2, 1+e_3$. Material fibres in these principal directions are stretched only, in which case the deformation is called a **pure deformation**; fibres in other directions will be stretched and rotated.

The term $\boldsymbol{\varepsilon} d\mathbf{X}$ in 2.7.6 therefore corresponds to a pure stretch along the principal axes. The total deformation is the sum of a pure deformation, represented by $\boldsymbol{\varepsilon}$, and a rigid body rotation, represented by $\boldsymbol{\Omega}$. This result is similar to that obtained for the exact finite strain theory, but here the decomposition is *additive* rather than *multiplicative*. Indeed, here the corresponding small strain stretch and rotation tensors are $\mathbf{U} = \mathbf{I} + \boldsymbol{\varepsilon}$ and $\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega}$, so that

$$\mathbf{F} = \mathbf{RU} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \qquad (2.7.7)$$

Example

Consider the simple shear (*c.f.* Eqn. 2.2.40)

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

where k is small. The displacement vector is $\mathbf{u} = kx_2\mathbf{e}_1$ so that

$$\operatorname{grad}\mathbf{u} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deformation can be written as the additive decomposition

$$d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\Omega} d\mathbf{X} \qquad \text{or} \qquad d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X}$$

with

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\boldsymbol{\omega} = -(k/2)\mathbf{e}_3$. For the rotation component, one can write

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega} = \begin{bmatrix} 1 & k/2 & 0 \\ -k/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, since for small θ, $\cos\theta \approx 1, \sin\theta \approx \theta$, can be seen to be a rotation through an angle $\theta = -k/2$ (a clockwise rotation).

The principal values of $\boldsymbol{\varepsilon}$ are $\pm k/2, 0$ with corresponding principal directions $\mathbf{n}_1 = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$, $\mathbf{n}_2 = -(1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$ and $\mathbf{n}_3 = \mathbf{e}_3$.

Thus the simple shear with small displacements consists of a rotation through an angle $k/2$ superimposed upon a pure shear with angle $k/2$, Fig. 2.6.2.

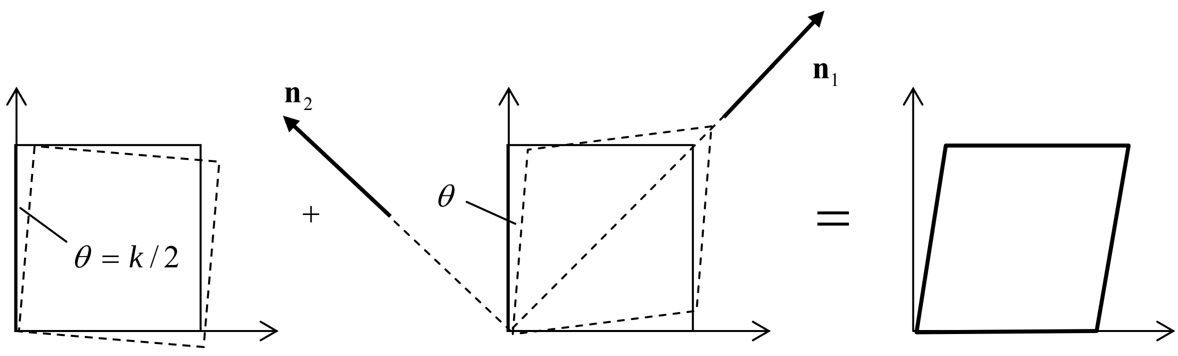


Figure 2.6.2: simple shear

■

2.7.2 Rotations and Small Strain

Consider now a pure rotation about the X_3 axis (within the exact finite strain theory), $d\mathbf{x} = \mathbf{R} d\mathbf{X}$, with

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.7.8)$$

This rotation does not change the length of line elements $d\mathbf{X}$. According to the small strain theory, however,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which does predict line element length changes, but which can be neglected if θ is small. For example, if the rotation is of the order $10^{-2}\,\text{rad}$, then $\varepsilon_{11} = \varepsilon_{22} = 10^{-4}$. However, if the rotation is large, the errors will be appreciable; in that case, rigid body rotation introduces geometrical non-linearities which must be dealt with using the finite deformation theory.

Thus the small strain theory is restricted to not only the case of small displacement gradients, but also small rigid body rotations.

2.7.3 Volume Change

An elemental cube with edges of unit length in the directions of the principal axes deforms into a cube with edges of lengths $1+e_1, 1+e_2, 1+e_3$, so the unit change in volume of the cube is

$$\frac{dv - dV}{dV} = (1+e_1)(1+e_2)(1+e_3) - 1 = e_1 + e_2 + e_3 + O(2) \qquad (2.7.9)$$

Since second order quantities have already been neglected in introducing the small strain tensor, they must be neglected here. Hence the increase in volume per unit volume, called the **dilatation** (or **dilation**) is

$$\boxed{\frac{\delta V}{V} = e_1 + e_2 + e_3 = e_{ii} = \text{tr}\,\boldsymbol{\varepsilon} = \text{div}\,\mathbf{u}} \qquad \textbf{Dilatation} \qquad (2.7.10)$$

Since any elemental volume can be constructed out of an infinite number of such elemental cubes, this result holds for any elemental volume irrespective of shape.

2.7.4 Rate of Deformation, Strain Rate and Spin Tensors

Take now the expressions 2.4.7 for the rate of deformation and spin tensors. Replacing **v** in these expressions by $\dot{\mathbf{u}}$, one has

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad d_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right)$$
$$\mathbf{w} = \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad w_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \frac{\partial \dot{u}_j}{\partial x_i}\right) \tag{2.7.11}$$

For small strains, one can take the time derivative outside (by considering the x_i to be material coordinates independent of time):

$$d_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}$$
$$w_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right\} \tag{2.7.12}$$

The rate of deformation in this context is seen to be the **rate of strain**, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, and the spin is seen to be the **rate of rotation**, $\mathbf{w} = \dot{\boldsymbol{\Omega}}$.

The instantaneous motion of a material particle can hence be regarded as the sum of three effects:

(i) a translation given by $\dot{\mathbf{u}}$ (so in the time interval Δt the particle has been displaced by $\dot{\mathbf{u}}\Delta t$)
(ii) a pure deformation given by $\dot{\boldsymbol{\varepsilon}}$
(iii) a rigid body rotation given by $\dot{\boldsymbol{\Omega}}$

2.7.5 Compatibility Conditions

Suppose that the strains ε_{ij} in a body are known. If the displacements are to be determined, then the strain-displacement partial differential equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.7.13}$$

need to be integrated. However, there are six independent strain components but only three displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**, and it can be shown that they are given by

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0 \tag{2.7.14}$$

These are 81 equations, but only six of them are distinct, and these six equations are necessary and sufficient to evaluate the displacement field.

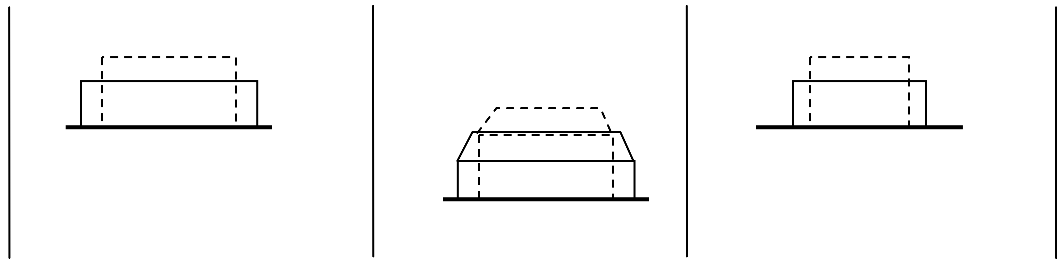
2.8 Objectivity and Objective Tensors

2.8.1 Dependence on Observer

Consider a rectangular block of material resting on a circular table. A person stands and observes the material deform, Fig. 2.8.1a. The dashed lines indicate the undeformed material whereas the solid line indicates the current state. A second observer is standing just behind the first, but on a step ladder – this observer sees the material as in 2.8.1b. A third observer is standing around the table, 45° from the first, and sees the material as in Fig. 2.8.1c.

The deformation can be described by each observer using concepts like displacement, velocity, strain and so on.. However, it is clear that the three observers will in general record different values for these measures, since their perspectives differ.

The goal in what follows is to determine which of the kinematical tensors are in fact *independent* of observer. Since the laws of physics describing the response of a deforming material must be independent of any observer, it is these particular tensors which will be more readily used in expressions to describe material response.



(a) (b) (c)

Figure 2.8.1: a deforming material as seen by different observers

Note that Fig. 2.8.1 can be interpreted in another, equivalent, way. One can imagine *one* static observer, but this time with the material moved into three different positions. This viewpoint will be returned to in the next section.

2.8.2 Change of Reference Frame

Consider two **frames of reference**, the first consisting of the origin **o** and the basis $\{\mathbf{e}_i\}$, the second consisting of the origin $\mathbf{o}^*$ and the basis $\{\mathbf{e}_i^*\}$, Fig. 2.8.2. A point **x** in space is then identified as having position vector $\mathbf{x} = x_i\mathbf{e}_i$ in the first frame and position vector $\mathbf{x}^* = x_i^*\mathbf{e}_i^*$ in the second frame.

When the origins **o** and $\mathbf{o}^*$ coincide, $\mathbf{x} = \mathbf{x}^*$ and the vector components x_i and x_i^* are related through Eqn. 1.5.3, $x_i = Q_{ij}x_j^*$, or $\mathbf{x} = x_i\mathbf{e}_i = Q_{ij}x_j^*\mathbf{e}_i$, where $[\mathbf{Q}]$ is the transformation matrix 1.5.4, $Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j^*$. Alternatively, one has Eqn. 1.5.5, $x_i^* = Q_{ji} x_j$, or $\mathbf{x}^* = x_i^* \mathbf{e}_i^* = Q_{ji} x_j \mathbf{e}_i^*$.

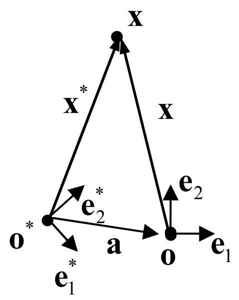


Figure 2.8.2: two frames of reference

With the shift in origin $\mathbf{a} = \mathbf{o} - \mathbf{o}^*$, one has

$$\mathbf{x}^* = x_i^* \mathbf{e}_i^* = Q_{ji} x_j \mathbf{e}_i^* + a_i^* \mathbf{e}_i^* \tag{2.8.1}$$

where $\mathbf{a} = a_i^* \mathbf{e}_i^*$. Alternatively,

$$\mathbf{x} = x_i \mathbf{e}_i = Q_{ij} x_j^* \mathbf{e}_i - a_i \mathbf{e}_i \tag{2.8.2}$$

where $\mathbf{a} = a_i \mathbf{e}_i$, with $a_i^* = Q_{ji} a_j$.

Formulae 2.8.1-2 relate the coordinates of the position vector to a point in space as observed from one frame of reference to the coordinates of the position vector to the *same* point as observed from a different frame of reference.

Finally, consider the position vector **x**, which is defined relative to the frame $(\mathbf{o}, \mathbf{e}_i)$. To an observer in the frame $(\mathbf{o}^*, \mathbf{e}_i^*)$, the *same* position vector would appear as $(\mathbf{x})^*$, Fig. 2.8.3. Rotating this vector $(\mathbf{x})^*$ through $\mathbf{Q}^{\mathrm{T}}$ (the tensor which rotates the basis $\{\mathbf{e}_i^*\}$ into the basis $\{\mathbf{e}_i\}$) and adding the vector **a** then produces $\mathbf{x}^*$:

$$\mathbf{x}^* = \mathbf{Q}^{\mathrm{T}} (\mathbf{x})^* + \mathbf{a} \tag{2.8.3}$$

This relation will be discussed further below.

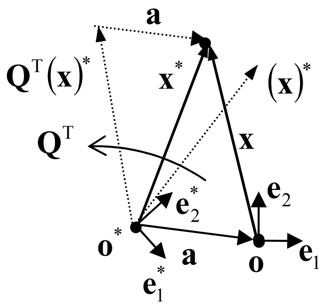


Figure 2.8.3: Relation between vectors in Eqn. 2.8.3

2.8.3 Change of Observer

The change of frame encompassed by Eqns. 2.8.1-2 is more precisely called a **passive change of frame**, and merely involves a transformation between vector components. One would say that there is one observer but that this observer is using two frames of reference. Here follows a different concept, an **active change of frame**, also called a **change in observer**, in which there are two observers, each with their own frame of reference.

An **observer** is someone who can measure relative positions in space (with a ruler) and instants of time (with a clock). An **event** in the physical world (for example a material particle) is perceived by an observer as occurring at a particular point in space and at a particular time. One can regard an observer O to be a map of an event E in the physical world to a point $\mathbf{x}$ in point space (*cf.* §1.2.5) and a real number t. A *single* event E is recorded as the pair $(\mathbf{x}, t)$ by an observer O and, in general, by a *different* pair $(\mathbf{x}^*, t^*)$ by a second observer O^*, Fig. 2.8.4.

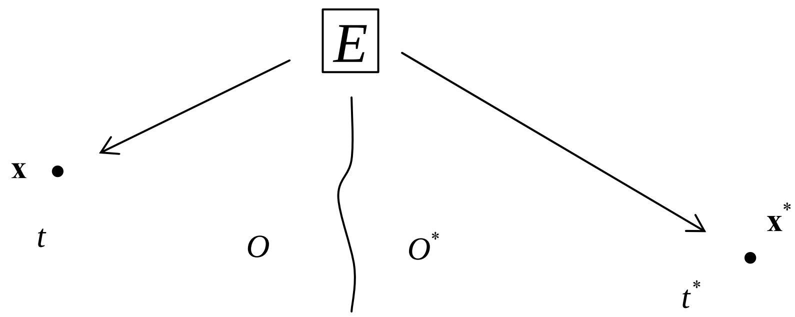


Figure 2.8.4: recordings by two observers of the same event

Let the two observers record three points corresponding to three events, Fig. 2.8.5. These points define vectors in space, as the difference between the points (*cf.* §1.2.5). It is assumed that both observers "see" the same Euclidean geometry, that is, if one observer sees an ellipse, then the other observer will see the same ellipse, but perhaps positioned differently in space. To ensure that this is so, observed vectors must be related through some orthogonal tensor $\mathbf{Q}$, for example,

$$\mathbf{x}^* - \mathbf{x}_0^* = \mathbf{Q}(\mathbf{x} - \mathbf{x}_0) \tag{2.8.4}$$

since this transformation will automatically preserve distances between points, and angles between vectors (see §1.10.7), for example,

$$\left(\mathbf{x}_1^* - \mathbf{x}_0^*\right)\cdot\left(\mathbf{x}^* - \mathbf{x}_0^*\right) = \mathbf{Q}\left(\mathbf{x}_1 - \mathbf{x}_0\right)\cdot\mathbf{Q}\left(\mathbf{x} - \mathbf{x}_0\right) = \left(\mathbf{x}_1 - \mathbf{x}_0\right)\cdot\left(\mathbf{x} - \mathbf{x}_0\right) \qquad (2.8.5)$$

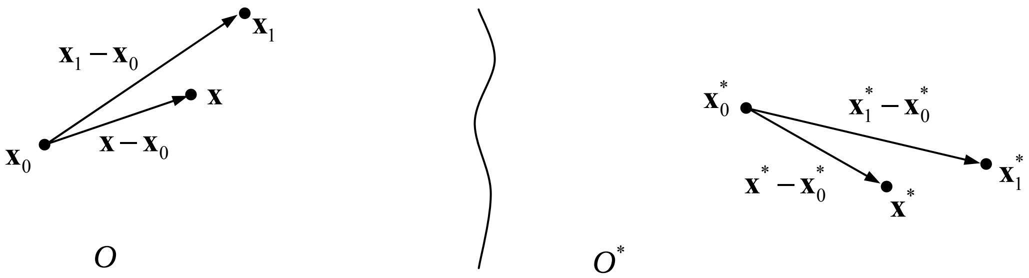


Figure 2.8.5: recordings of two observers of three separate events

Although all orthogonal tensors $\mathbf{Q}$ do indeed preserve length and angles, it is taken that the $\mathbf{Q}$ in 2.8.4-5 is proper orthogonal, i.e. a rotation tensor (*cf.* §1.10.8), so that orientation is also preserved. Further, it is assumed that $\mathbf{Q} = \mathbf{Q}(t)$, which expresses the fact that the observers can move relative to each other over time.

Observers must also agree on time intervals between events. Let an observer *O* record a certain event at time *t* and a second observer O^* record the same event as occurring at time t^*. Then the times must be related through

$$\boxed{t^* = t + \alpha} \quad \textbf{Observer Time Transformation} \qquad (2.8.6)$$

where α is a *constant*. If now the observers record a second event as occurring at t_1 and t_1^* say, one has $t_1^* - t^* = t_1 - t$ as required.

The observer transformation 2.8.4 involves the vectors $\mathbf{x} - \mathbf{x}_0$ and $\mathbf{x}^* - \mathbf{x}_0^*$ and as such does not require the notion of origin or coordinate system; it is an abstract symbolic notation for an observer transformation. However, an origin $\mathbf{o}$ for O and $\mathbf{o}^*$ for O^* can be introduced and then the points $\mathbf{x}_0$, $\mathbf{x}$, $\mathbf{x}_0^*$, $\mathbf{x}^*$ can be regarded as *position vectors* in space, Fig. 2.8.6.

The transformation 2.8.4 can now be expressed in the oft-used format

$$\boxed{\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}} \quad \textbf{Observer (Spatial) Transformation} \qquad (2.8.7)$$

where

$$\mathbf{c}(t) = \mathbf{x}_0^* - \mathbf{Q}(t)\mathbf{x}_0 \qquad (2.8.8)$$

The transformation 2.8.7 is called a **Euclidean transformation**, since it preserves the Euclidean geometry.

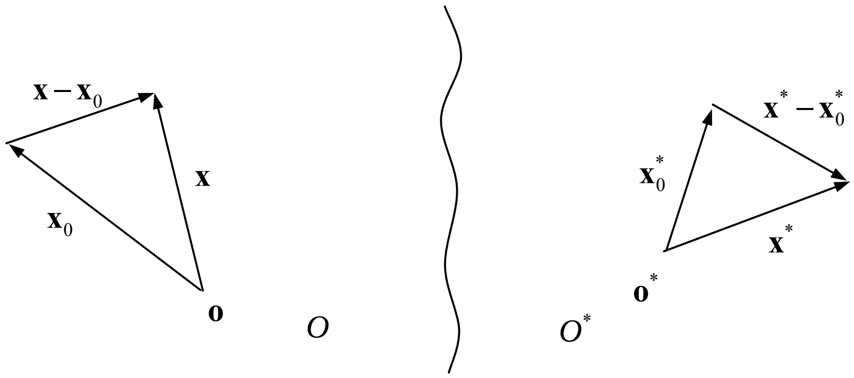


Figure 2.8.6: position vectors for two observers of the same events

Coordinate Systems

Each observer can introduce any Cartesian coordinate system, with basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i^*\}$ say. They can then resolve the position vectors into vector components. These basis vectors can be oriented with respect to each other in any way, that is, they will be related through $\mathbf{e}_i^* = \mathbf{R}\mathbf{e}_i$, where $\mathbf{R}$ is any rotation tensor. Indeed, each observer can change their basis, effecting a coordinate transformation. No attempt to introduce specific coordinate systems will be made here since they are completely unnecessary to the notion of observer transformation and would only greatly confuse the issue.

Relationship to Passive Change of Frame

Recall the passive change of frame encompassed in Eqns. 2.8.1-2. If one substitutes the *actual* $\mathbf{x}$ for $(\mathbf{x})^*$ in Eqn. 2.8.3, one has:

$$\mathbf{x}^* = \mathbf{Q}^{\mathrm{T}}\mathbf{x} + \mathbf{a} \tag{2.8.9}$$

This is clearly an observer transformation, relating the position vector as seen by one observer to the position vector as seen by a second observer, through an orthogonal tensor and a vector, as in Eqn. 2.8.7. In the passive change of frame, Q_{ij} are the components of the orthogonal tensor $\mathbf{Q} = \mathbf{e}_i^* \otimes \mathbf{e}_i$, Eqn. 1.10.25, which maps the bases onto each other: $\mathbf{e}_i^* = \mathbf{Q}\mathbf{e}_i$. Thus the transformation 2.8.1-2 can be defined uniquely by the pair $\mathbf{Q}$ and $\mathbf{a}$. In that sense, the passive change of frame does indeed define an active change of frame, i.e. a change of observer, through Eqn. 2.8.9. However, the concept of observer discussed above is the preferred way of defining an observer transformation.

2.8.4 Objective Vectors and Tensors

The observer transformation 2.8.7 encapsulates the different viewpoints observers have of the physical world. They will see the same objects, but in general they will see these objects oriented differently and located at different positions. The goal now is to see which of the kinematical tensors are independent of these different viewpoints. As a first step, next is introduced the concept of an **objective tensor**.

Suppose that different observers are examining a deforming material. In order to describe the material, the observers take measurements. This will involve measurements of *spatial* objects associated with the current configuration, for example the velocity or spin. It will also involve *material* objects associated with the reference configuration, for example line elements in that configuration. It will also involve *two-point* tensors such as the rotation or deformation gradient, which are associated with both the current and reference configurations.

It is assumed that all observers observe the reference configuration to be the same, that is, they record the same set of points for the material particles in the reference configuration[1]. The observers then move relative to each other and their measurements of objects associated with the current configuration will in general differ. One would expect (want) different observers to make the same measurement of material objects despite this relative movement; thus one says that material vectors and tensors are **objective (material) vectors** and **objective (material) tensors** if they remain unchanged under the observer transformation 2.8.6-7.

A spatial vector $\mathbf{u}$ on the other hand is said to be an **objective (spatial) vector** if it satisfies the observer transformation (see 2.8.4):[2]

$$\boxed{\mathbf{u}^* = \mathbf{Qu}} \qquad \textbf{Objectivity Requirement for a Spatial Vector} \qquad (2.8.10)$$

for all rotation tensors $\mathbf{Q}$. An **objective (spatial) tensor** is defined to be one which transforms an objective vector into an objective vector. Consider a tensor observed as $\mathbf{T}$ and $\mathbf{T}^*$ by two different observers. Take an objective vector which is observed as $\mathbf{v}$ and $\mathbf{v}^*$, and let $\mathbf{u} = \mathbf{Tv}$ and $\mathbf{u}^* = \mathbf{T}^*\mathbf{v}^*$. Then, for $\mathbf{u}$ to be objective,

$$\mathbf{u}^* = \mathbf{Qu} = \mathbf{QTv} = \mathbf{QTQ}^{\mathrm{T}}\mathbf{v}^* \qquad (2.8.11)$$

and so the tensor is objective provided

$$\boxed{\mathbf{T}^* = \mathbf{QTQ}^{\mathrm{T}}} \qquad \textbf{Objectivity Requirement for a Spatial Tensor} \qquad (2.8.12)$$

Various identities can be derived; for example, for objective vectors $\mathbf{a}$ and $\mathbf{b}$, and objective tensors $\mathbf{A}$ and $\mathbf{B}$, {▲Problem 1}

[1] this does not affect the generality of what follows; the notion of objective tensor is independent of the chosen reference configuration
[2] the time transformation 2.8.6 is trivial and does not affect the relations to be derived

$$
\begin{aligned}
(\mathbf{a}+\mathbf{b})^* &= \mathbf{a}^* + \mathbf{b}^* \\
(\mathbf{a}\otimes\mathbf{b})^* &= \mathbf{a}^* \otimes \mathbf{b}^* \\
(\mathbf{a}\cdot\mathbf{b})^* &= \mathbf{a}^* \cdot \mathbf{b}^* \\
(\mathbf{A}\mathbf{b})^* &= \mathbf{A}^*\mathbf{b}^* \\
(\mathbf{A}\mathbf{B})^* &= \mathbf{A}^*\mathbf{B}^* \\
(\mathbf{A}^{-1})^* &= (\mathbf{A}^*)^{-1} \\
(\mathbf{A}\mathbf{B})^* &= \mathbf{A}^*\mathbf{B}^* \\
(\mathbf{A}:\mathbf{B})^* &= \mathbf{A}^* : \mathbf{B}^*
\end{aligned}
\qquad (2.8.13)
$$

For a scalar,

$$\boxed{\phi^* = \phi} \qquad \textbf{Objectivity Requirement for a Scalar} \qquad (2.8.14)$$

In other words, an **objective scalar** is one which has the same value to all observers.

Finally, consider a two-point tensor. Such a tensor is said to be objective if it maps an objective material vector into an objective spatial vector. Consider then a two-point tensor observed as $\mathbf{T}$ and $\mathbf{T}^*$. Take an objective material vector which is observed as $\mathbf{v}$ and $\mathbf{v}^*$, and let $\mathbf{u} = \mathbf{T}\mathbf{v}$ and $\mathbf{u}^* = \mathbf{T}^*\mathbf{v}^*$. A material vector is objective if it is unaffected by an observer transformation, so

$$\mathbf{u}^* = \mathbf{Q}\mathbf{u} = \mathbf{Q}\mathbf{T}\mathbf{v} = \mathbf{Q}\mathbf{T}\mathbf{v}^* \qquad (2.8.15)$$

and so the tensor is objective provided

$$\boxed{\mathbf{T}^* = \mathbf{Q}\mathbf{T}} \qquad \textbf{Objectivity Requirement for a Two-point Tensor} \qquad (2.8.16)$$

Thus the objectivity requirement for a two-point tensor is the same as that for a spatial vector.

2.8.5 Objective Kinematics

Next are examined the various kinematic vectors and tensors introduced in the earlier sections, and their objectivity status is determined.

The motion is observed by one observer as $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ and by a second observer as $\mathbf{x}^* = \boldsymbol{\chi}(\mathbf{X}, t^*)$. The observer transformation gives

$$\boldsymbol{\chi}^*(\mathbf{X}, t^*) = \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{c}(t), \qquad t^* = t + \alpha \qquad (2.8.17)$$

and so the motion is not an objective vector, i.e. $\boldsymbol{\chi}^* \neq \mathbf{Q}\boldsymbol{\chi}$.

The Velocity and Acceleration

Differentiating 2.8.17 (and using the notation $\dot{\mathbf{x}}$ instead of $\dot{\boldsymbol{\chi}}(\mathbf{X},t)$ for brevity), the velocity under the observer transformation is

$$\dot{\mathbf{x}}^* = \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\dot{\mathbf{x}} + \dot{\mathbf{c}} \tag{2.8.18}$$

which does not comply with the objectivity requirement for spatial vectors, 2.8.10. In other words, different observers will measure different magnitudes for the velocity. The velocity expression can be put in a form similar to that of elementary mechanics (the "non-objective" terms are on the right),

$$\dot{\mathbf{x}}^* - \mathbf{Q}\dot{\mathbf{x}} = \mathbf{\Omega}_\mathrm{Q}\left(\mathbf{x}^* - \mathbf{c}\right) + \dot{\mathbf{c}} \tag{2.8.19}$$

where

$$\mathbf{\Omega}_\mathrm{Q} = \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} \tag{2.8.20}$$

is skew-symmetric (see Eqn. 1.14.2); this tensor represents the rigid body angular velocity between the observers (see Eqn. 2.6.1). Note that the velocity *is* objective provided $\dot{\mathbf{Q}} = \mathbf{0}$, $\dot{\mathbf{c}} = \mathbf{o}$, for which $\mathbf{x}^* = \mathbf{Q}_0\mathbf{x} + \mathbf{c}_0$, which is called a **time-independent rigid transformation**.

Similarly, for the acceleration, it can be shown that

$$\ddot{\mathbf{x}}^* - \mathbf{Q}\ddot{\mathbf{x}} = \dot{\mathbf{\Omega}}_\mathrm{Q}\left(\mathbf{x}^* - \mathbf{c}\right) - \mathbf{\Omega}_\mathrm{Q}^2\left(\mathbf{x}^* - \mathbf{c}\right) + 2\mathbf{\Omega}_\mathrm{Q}\left(\dot{\mathbf{x}} - \dot{\mathbf{c}}\right) + \ddot{\mathbf{c}} \tag{2.8.21}$$

The first three terms on the right-hand side are called the **Euler acceleration**, the **centrifugal acceleration** and the **Coriolis acceleration** respectively. The acceleration is objective provided $\dot{\mathbf{c}}$ and $\mathbf{Q}$ are constant, for which $\mathbf{x}^* = \mathbf{Q}_0\mathbf{x} + \mathbf{c}(t)$ with $\ddot{\mathbf{c}} = \mathbf{o}$, which is called a **Galilean transformation** – where the two configurations are related by a rigid rotation and a translational motion with constant velocity.

The Deformation Gradient

Consider the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$. As mentioned, observers observe the reference configuration to be the same: $\mathbf{X}^* = \mathbf{X}$. The deformation is then observed as $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ and $d\mathbf{x}^* = \mathbf{F}^*d\mathbf{X}$, so that

$$d\mathbf{x}^* = \mathbf{Q}d\mathbf{x} = \mathbf{QF}d\mathbf{X} = \mathbf{QF}d\mathbf{X} \tag{2.8.22}$$

and

$$\mathbf{F}^* = \mathbf{QF} \tag{2.8.23}$$

and so, according to 2.8.16, the deformation gradient is objective.

The Cauchy-Green Strain Tensors

For the right and left Cauchy-Green tensors,

$$\begin{aligned} \mathbf{C}^* &= \mathbf{F}^{*\mathrm{T}}\mathbf{F}^* = \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{F} = \mathbf{C} \\ \mathbf{b}^* &= \mathbf{F}^*\mathbf{F}^{*\mathrm{T}} = \mathbf{Q}\mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}} \end{aligned} \tag{2.8.24}$$

Thus the material tensor **C** and the spatial tensor **b** are objective[3].

The Jacobian Determinant

For the Jacobian determinant, using 1.10.16a,

$$J^* = \det \mathbf{F}^* = \det(\mathbf{Q}\mathbf{F}) = \det \mathbf{Q} \det \mathbf{F} = \det \mathbf{F} = J \tag{2.8.25}$$

and[4] so is objective according to 2.8.14.

The Rotation and Stretch Tensors

The polar decomposition is $\mathbf{F} = \mathbf{R}\mathbf{U}$, where **R** is the orthogonal rotation tensor and **U** is the right stretch tensor. Then $\mathbf{F}^* = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{R}\mathbf{U} \equiv \mathbf{R}^*\mathbf{U}^*$. Since $\mathbf{Q}\mathbf{R}$ is orthogonal, the expression $\mathbf{Q}\mathbf{R}\mathbf{U} = \mathbf{R}^*\mathbf{U}^*$ is valid provided

$$\mathbf{R}^* = \mathbf{Q}\mathbf{R}, \qquad \mathbf{U}^* = \mathbf{U} \tag{2.8.26}$$

Thus the two-point tensor **R** and the material tensor **U** are objective.

The Velocity Gradient

Allowing **Q** to be a function of time, for the velocity gradient, using 2.5.4, 1.9.18c,

$$\mathbf{l}^* = \dot{\overline{\mathbf{F}^*}}(\mathbf{F}^*)^{-1} = (\mathbf{Q}\dot{\mathbf{F}} + \dot{\mathbf{Q}}\mathbf{F})\mathbf{F}^{-1}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}} + \mathbf{\Omega}_{\mathbf{Q}} \tag{2.8.27}$$

where $\mathbf{\Omega}_{\mathbf{Q}}$ is the angular velocity tensor 2.8.20. On the other hand, with $\mathbf{l} = \mathbf{d} + \mathbf{w}$, and separating out the symmetric and skew-symmetric parts,

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{w}^* = \mathbf{Q}\mathbf{w}\mathbf{Q}^{\mathrm{T}} + \mathbf{\Omega}_{\mathbf{Q}} \tag{2.8.28}$$

Thus the velocity gradient is not objective. This is not surprising given that the velocity is not objective. However, significantly, the rate of deformation, a measure of the rate of stretching of material, *is* objective.

---

[3] Some authors define a second order tensor to be objective only if 2.8.12 is satisfied, regardless of whether it is spatial, two-point or material; with this definition, **F** and **C** would be defined as non-objective
[4] Note that **Q** must be a rotation tensor, not just an orthogonal tensor, here

The Spatial Gradient

Consider the spatial gradient of an *objective vector* **t**:

$$\operatorname{grad}\mathbf{t} = \frac{\partial \mathbf{t}}{\partial \mathbf{x}}, \qquad (\operatorname{grad}\mathbf{t})^* = \frac{\partial \mathbf{t}^*}{\partial \mathbf{x}^*} \tag{2.8.29}$$

Since $\mathbf{t}^* = \mathbf{Qt}$, the chain rule gives

$$\frac{\partial \mathbf{t}^*}{\partial \mathbf{x}} = \frac{\partial \mathbf{t}^*}{\partial \mathbf{x}^*}\frac{\partial \mathbf{x}^*}{\partial \mathbf{x}} \equiv \frac{\partial (\mathbf{Qt})}{\partial \mathbf{x}} = \mathbf{Q}\frac{\partial \mathbf{t}}{\partial \mathbf{x}} \tag{2.8.30}$$

It follows that

$$(\operatorname{grad}\mathbf{t})^* = \mathbf{Q}\frac{\partial \mathbf{t}}{\partial \mathbf{x}}\mathbf{Q}^{\mathrm{T}} \tag{2.8.31}$$

Thus the spatial gradient is objective. In general, it can be shown that the spatial gradient of a tensor field of order *n* is objective, for example the gradient of a scalar ϕ, {▲Problem 2} $\operatorname{grad}\phi$. Further, for a vector **v**, {▲Problem 3} $\operatorname{div}\mathbf{v}$ is objective.

Objective Rates

Consider an objective vector field **u**. The material derivative $\dot{\mathbf{u}}$ is not objective.

However, the co-rotational derivative, Eqn. 2.6.12, $\overset{\circ}{\mathbf{u}} = \dot{\mathbf{u}} - \mathbf{wu}$ *is* objective. To show this, contract 2.8.28b, $\mathbf{w}^* = \mathbf{QwQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$, to the right with $\mathbf{Q}$ to get an expression for $\dot{\mathbf{Q}}$:

$$\dot{\mathbf{Q}} = \mathbf{w}^*\mathbf{Q} - \mathbf{Qw} \tag{2.8.32}$$

and then

$$\mathbf{u}^* = \mathbf{Qu} \quad \to \quad \dot{\overline{\mathbf{u}^*}} = \dot{\mathbf{Q}}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}} = \mathbf{w}^*\mathbf{Qu} + \mathbf{Q}(\dot{\mathbf{u}} - \mathbf{wu}) = \mathbf{w}^*\mathbf{Qu} + \mathbf{Q}\overset{\circ}{\mathbf{u}} \tag{2.8.33}$$

Then $\dot{\overline{\mathbf{u}^*}} - \mathbf{w}^*\mathbf{u}^* = \mathbf{Q}\overset{\circ}{\mathbf{u}}$, or $(\overset{\circ}{\mathbf{u}})^* = \mathbf{Q}\overset{\circ}{\mathbf{u}}$, so that the co-rotational derivative of a vector is an objective vector.

Rates of spatial tensors can also be modified in order to construct objective rates. For example, consider an objective spatial tensor **T**, so $\mathbf{T}^* = \mathbf{QTQ}^{\mathrm{T}}$. Then

$$\dot{\overline{\mathbf{T}^*}} = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{TQ}^{\mathrm{T}} + \mathbf{QT}\dot{\mathbf{Q}}^{\mathrm{T}} \tag{2.8.34}$$

which is clearly not objective. However, this can be re-arranged using 2.8.32 into

$$\overline{\mathbf{T}^*}^{\cdot} - \mathbf{w}^*\mathbf{T}^* + \mathbf{T}^*\mathbf{w}^* = \mathbf{Q}\left(\dot{\mathbf{T}} - \mathbf{w}\mathbf{T} + \mathbf{T}\mathbf{w}\right)\mathbf{Q}^{\mathrm{T}} \tag{2.8.35}$$

and so the quantity

$$\dot{\mathbf{T}} - \mathbf{w}\mathbf{T} + \mathbf{T}\mathbf{w} \tag{2.8.36}$$

is an objective rate, called the **Jaumann rate**. Other objective rates of tensors can be constructed in a similar fashion, for example the **Cotter-Rivlin rate**, defined by {▲Problem 4}

$$\dot{\mathbf{T}} + \mathbf{l}^{\mathrm{T}}\mathbf{T} + \mathbf{T}\mathbf{l} \tag{2.8.37}$$

Summary of Objective Kinematic Objects

Table 2.8.1 summarises the objectivity of some important kinematic objects:

	objective	definition	Type	Transformation
Jacobian determinant	✓		Scalar	$J^* = J$
Deformation gradient	✓		2-point	$\mathbf{F}^* = \mathbf{QF}$
Rotation	✓	$\mathbf{R} = \mathbf{FU}^{-1} = \mathbf{v}^{-1}\mathbf{F}$	2-point	$\mathbf{R}^* = \mathbf{QR}$
Right Cauchy-Green strain	✓	$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	Material	$\mathbf{C}^* = \mathbf{C}$
Green-Lagrange strain	✓	$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	Material	$\mathbf{E}^* = \mathbf{E}$
Rate of Green-Lagrange strain	✓		Material	$\overline{\mathbf{E}^*}^{\cdot} = \dot{\mathbf{E}}$
Right Stretch	✓	$\mathbf{U} = \sqrt{\mathbf{C}}$	Material	$\mathbf{U}^* = \mathbf{U}$
Left Cauchy-Green strain	✓	$\mathbf{b} = \mathbf{FF}^{\mathrm{T}}$	Spatial	$\mathbf{b}^* = \mathbf{QbQ}^{\mathrm{T}}$
Euler-Almansi strain	✓	$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$	Spatial	$\mathbf{e}^* = \mathbf{QeQ}^{\mathrm{T}}$
Left Stretch	✓	$\mathbf{v} = \sqrt{\mathbf{b}}$	Spatial	$\mathbf{v}^* = \mathbf{QvQ}^{\mathrm{T}}$
Spatial Velocity Gradient	×	$\mathbf{l} = \mathrm{grad}\,\mathbf{v}$	Spatial	$\mathbf{l}^* = \mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$
Rate of Deformation	✓	$\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^{\mathrm{T}})$	Spatial	$\mathbf{d}^* = \mathbf{QdQ}^{\mathrm{T}}$
Spin	×	$\mathbf{w} = \frac{1}{2}(\mathbf{l} - \mathbf{l}^{\mathrm{T}})$	Spatial	$\mathbf{w}^* = \mathbf{QwQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$

Table 2.8.1: Objective kinematic objects

2.8.6 Objective Functions

In a similar way, functions are defined to be objective as follows:

- A scalar-valued function ϕ of, for example, a tensor **A**, is objective if it transforms in the same way as an objective scalar,

$$\phi^*(\mathbf{A}) = \phi(\mathbf{A}) \tag{2.8.38}$$

- A (spatial) vector-valued function **a** of a tensor **A** is objective if it transforms in the same way as an objective vector

$$\mathbf{v}^*(\mathbf{A}) = \mathbf{Q}\mathbf{v}(\mathbf{A}) \tag{2.8.39}$$

- A (spatial) tensor-valued function **f** of a tensor **A** is objective if it transforms according to

$$\mathbf{f}^*(\mathbf{A}) = \mathbf{Q}\mathbf{f}(\mathbf{A})\mathbf{Q}^{\mathrm{T}} \tag{2.8.40}$$

Objective functions of the Deformation Gradient

Consider an objective scalar-valued function ϕ of the deformation gradient **F**, $\phi(\mathbf{F})$. The function is objective if $\phi^* = \phi(\mathbf{F})$. But also,

$$\phi^* = \phi(\mathbf{F}^*) = \phi(\mathbf{QF}) \tag{2.8.41}$$

Using the polar decomposition theorem, $\phi(\mathbf{RU}) = \phi(\mathbf{QRU})$. Choosing the particular rigid-body rotation $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ then leads to

$$\phi(\mathbf{RU}) = \phi(\mathbf{U}) \tag{2.8.42}$$

which leads to the **reduced form**

$$\phi(\mathbf{F}) = \phi(\mathbf{U}) \tag{2.8.43}$$

Thus for the scalar function ϕ to be objective, it must be independent of the rotational part of **F**, and depends only on the stretching part; it cannot be a function of the nine independent components of the deformation gradient, but only of the six independent components of the right stretch tensor.

Consider next an objective (spatial) tensor-valued function **f** of the deformation gradient **F**, $\mathbf{f}(\mathbf{F})$. According to the definition of objectivity of a second order tensor, 2.8.12:

$$\mathbf{f}^* = \mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \tag{2.8.44}$$

But also,

$$\mathbf{f}^* = \mathbf{f}(\mathbf{F}^*) = \mathbf{f}(\mathbf{QF}) \tag{2.8.45}$$

Again, using the polar decomposition theorem and choosing the particular rigid-body rotation $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ leads to

$$\mathbf{f}(\mathbf{U}) = \mathbf{R}^{\mathrm{T}}\mathbf{f}(\mathbf{RU})\mathbf{R} \tag{2.8.46}$$

which leads to the reduced form

$$\mathbf{f}(\mathbf{F}) = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \tag{2.8.47}$$

Thus for **f** to be objective, its dependence on **F** must be through an arbitrary function of **U** together with a more explicit dependence on **R**, the rotation tensor

Example

Consider the tensor function $\mathbf{f}(\mathbf{F}) = \alpha\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^2$. Then

$$\mathbf{f}(\mathbf{Q}\mathbf{F}) = \alpha\left[(\mathbf{Q}\mathbf{F})(\mathbf{Q}\mathbf{F})^{\mathrm{T}}\right]^2 = \mathbf{Q}\alpha\left[\mathbf{F}\mathbf{F}^{\mathrm{T}}\right]^2\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}$$

and so the objectivity requirement is satisfied. According to the above, then, one can evaluate $\mathbf{f}(\mathbf{U}) = \mathbf{R}^{\mathrm{T}}\mathbf{f}(\mathbf{R}\mathbf{U})\mathbf{R} = \alpha\left(\mathbf{U}\mathbf{U}^{\mathrm{T}}\right)^2$, and the reduced form is

$$\mathbf{f} = \mathbf{R}\alpha\left(\mathbf{U}\mathbf{U}^{\mathrm{T}}\right)^2\mathbf{R}^{\mathrm{T}} = \alpha\mathbf{R}\mathbf{U}^4\mathbf{R}^{\mathrm{T}}$$

Also, since $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, alternative reduced forms are

$$\mathbf{f} = \mathbf{R}\mathbf{f}_2(\mathbf{C})\mathbf{R}^{\mathrm{T}}, \quad \mathbf{f} = \mathbf{R}\mathbf{f}_3(\mathbf{E})\mathbf{R}^{\mathrm{T}}$$

■

Finally, consider a *spatial* tensor function **f** of a *material* tensor **T**. Then

$$\mathbf{f}^*(\mathbf{T}) = \mathbf{Q}\mathbf{f}(\mathbf{T})\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{f}^*(\mathbf{T}) = \mathbf{f}(\mathbf{T}^*) = \mathbf{f}(\mathbf{T}) \tag{2.8.48}$$

It follows that

$$\mathbf{f} = \mathbf{Q}\mathbf{f}\mathbf{Q}^{\mathrm{T}} \tag{2.8.49}$$

This is true only in the special case $\mathbf{Q} = \mathbf{I}$ and so is not true in general. It follows that the function **f** is not objective.

2.8.7 Problems

1. Derive the relations 2.8.13
2. Show that the spatial gradient of a scalar ϕ is objective.
3. Show that the divergence of a spatial vector **v** is objective. [Hint: use the definition 1.11.9 and identity 1.9.10e]
4. Verify that the Rivlin-Cotter rate of a tensor **T**, $\mathbf{T} + \mathbf{l}^{\mathrm{T}}\mathbf{T} + \mathbf{T}\mathbf{l}$, is objective.

2.9 Rigid Body Rotations of Configurations

In this section are discussed rigid body rotations to the current and reference configurations.

2.9.1 A Rigid Body Rotation of the Current Configuration

As mentioned in §2.8.1, the circumstance of two observers, moving relative to each other and examining a fixed configuration (the current configuration) is equivalent to one observer taking measurements of two different configurations, moving relative to each other[1]. The objectivity requirements of the various kinematic objects discussed in the previous section can thus also be examined by considering rigid body rotations and translations of the current configuration.

Any rigid body rotation and translation of the current configuration can be expressed in the form

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}(t)\mathbf{x}(\mathbf{X},t) + \mathbf{c}(t) \tag{2.9.1}$$

where $\mathbf{Q}$ is a rotation tensor. This is illustrated in Fig. 2.9.5. The current configuration is denoted by S and the rotated configuration by S^*.

Just as $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, the deformation gradient for the configuration S^* relative to the reference configuration S_0 is defined through $d\mathbf{x}^* = \mathbf{F}^* d\mathbf{X}$. From 2.9.1, as in §2.8.5 (see Eqn. 2.8.23), and similarly for the right and left Cauchy-Green tensors,

$$\begin{aligned}
\mathbf{F}^* &= \mathbf{QF} \\
\mathbf{C}^* &= \mathbf{F}^{*\mathrm{T}}\mathbf{F}^* = \mathbf{C} \\
\mathbf{b}^* &= \mathbf{F}^*\mathbf{F}^{*\mathrm{T}} = \mathbf{QbQ}^{\mathrm{T}}
\end{aligned} \tag{2.9.2}$$

Thus in the deformations $\mathbf{F}: S_0 \to S$ and $\mathbf{F}^*: S_0 \to S^*$, the right Cauchy Green tensors, $\mathbf{C}$ and $\mathbf{C}^*$, are the same, but the left Cauchy Green tensors are different, and related through $\mathbf{b}^* = \mathbf{QbQ}^{\mathrm{T}}$.

All the other results obtained in the last section in the context of observer transformations, for example for the Jacobian, stretch tensors, etc., hold also for the case of rotations to the current configuration.

---

[1] Although equivalent, there is a difference: in one, there are two observers who record one event (a material particle say) as at two different points, in the other there is one observer who records two different events (the place where the one material particle is in two different configurations)

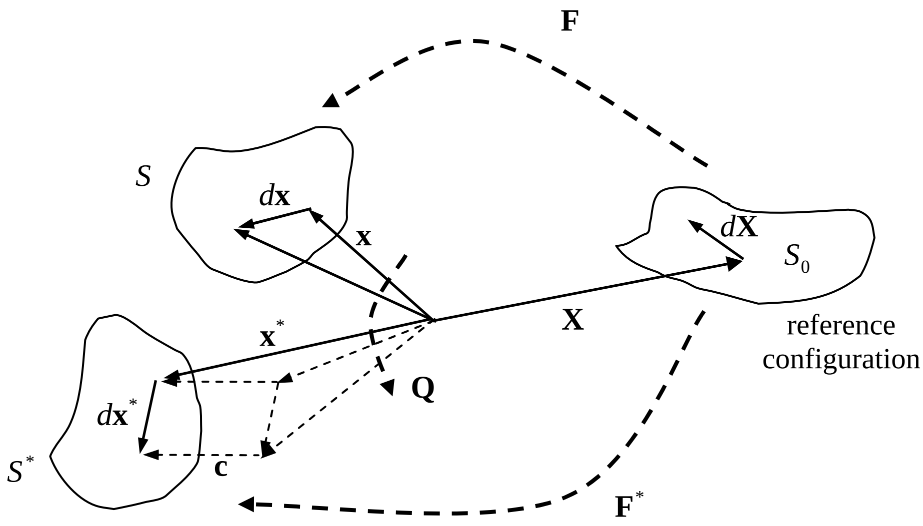


Figure 2.9.1: a rigid body rotation and translation of the current configuration

2.9.2 A Rigid Body Rotation of the Reference Configuration

Consider now a rigid-body rotation to the *reference* configuration. Such rotations play an important role in the notion of material symmetry (see Chapter 5).

The reference configuration is denoted by S_0 and the rotated/translated configuration by $S^{\Diamond}$, Fig. 2.9.2. The deformation gradient for the current configuration S relative to $S^{\Diamond}$ is defined through $d\mathbf{x} = \mathbf{F}^{\Diamond} d\mathbf{X}^{\Diamond} = \mathbf{F}^{\Diamond}\mathbf{Q}d\mathbf{X}$. But $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ and so (and similarly for the right and left Cauchy-Green tensors)

$$\begin{aligned}\mathbf{F}^{\Diamond} &= \mathbf{F}\mathbf{Q}^{\mathrm{T}} \\ \mathbf{C}^{\Diamond} &= \mathbf{F}^{\Diamond\mathrm{T}}\mathbf{F}^{\Diamond} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}} \\ \mathbf{b}^{\Diamond} &= \mathbf{F}^{\Diamond}\mathbf{F}^{\Diamond\mathrm{T}} = \mathbf{b}\end{aligned} \tag{2.9.3}$$

Thus the change to the right (left) Cauchy-Green strain tensor under a rotation to the reference configuration is the same as the change to the left (right) Cauchy-Green strain tensor under a rotation of the current configuration.

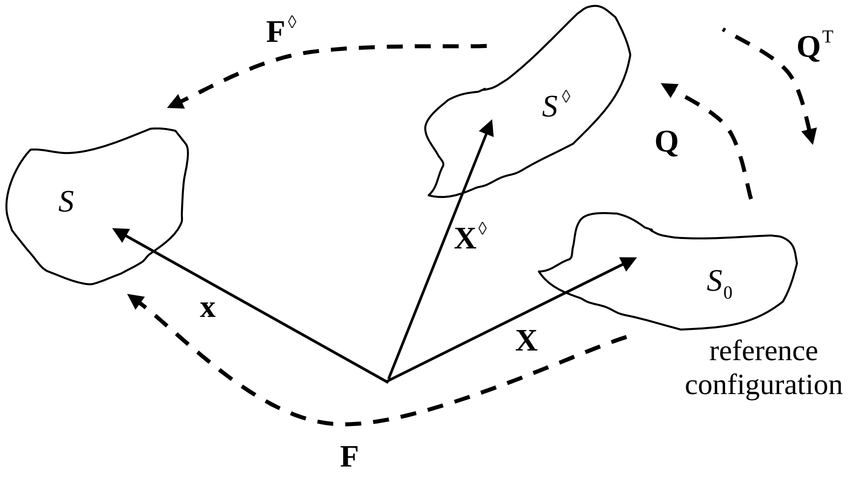


Figure 2.9.2: a rigid body rotation of the reference configuration

2.10 Convected Coordinates

An introduction to curvilinear coordinate was given in section 1.16, which serves as an introduction to this section. As mentioned there, the formulation of almost all mechanics problems, and their numerical implementation and solution, can be achieved using a description of the problem in terms of Cartesian coordinates. However, use of curvilinear coordinates allows for a deeper insight into a number of important concepts and aspects of, in particular, large strain mechanics problems. These include the notions of the Push Forward operation, Lie derivatives and objective rates.

As will become clear, note that all the tensor relations expressed in symbolic notation already discussed, such as $\mathbf{U} = \sqrt{\mathbf{C}}$, $\mathbf{F}\hat{\mathbf{N}}_i = \lambda_i \mathbf{n}_i$, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, etc., are independent of coordinate system, and hold also for the convected coordinates discussed here.

2.10.1 Convected Coordinates

In the Cartesian system, orthogonal coordinates X^i, x^i were used. Here, introduce the curvilinear coordinates Θ^i. The material coordinates can then be written as

$$\mathbf{X} = \mathbf{X}(\Theta^1, \Theta^2, \Theta^3) \tag{2.10.1}$$

so $\mathbf{X} = X^i \mathbf{E}_i$ and

$$d\mathbf{X} = dX^i \mathbf{E}_i = d\Theta^i \mathbf{G}_i \,, \tag{2.10.2}$$

where $\mathbf{G}_i$ are the covariant base vectors in the reference configuration, with corresponding contravariant base vectors $\mathbf{G}^i$, Fig. 2.10.1, with

$$\mathbf{G}^i \cdot \mathbf{G}_j = \delta^i_j \tag{2.10.3}$$

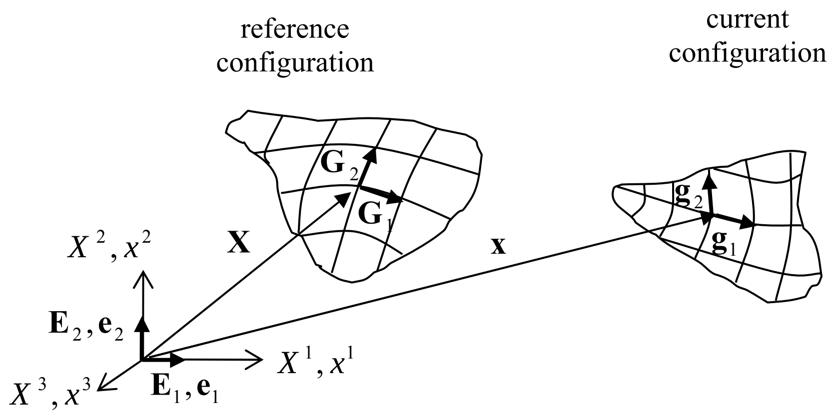


Figure 2.10.1: Curvilinear Coordinates

The coordinate curves form a net in the undeformed configuration (over the surfaces of constant Θ^i). One says that the curvilinear coordinates are **convected** or **embedded**, that is, the coordinate curves are attached to material particles and deform with the body, so that each material particle *has the same values* of the coordinates Θ^i in both the reference and current configurations. The covariant base vectors are tangent the coordinate curves.

In the current configuration, the spatial coordinates can be expressed in terms of a new, "current", set of curvilinear coordinates

$$\mathbf{x} = \mathbf{x}(\Theta^1, \Theta^2, \Theta^3, t), \tag{2.10.4}$$

with corresponding covariant base vectors $\mathbf{g}_i$ and contravariant base vectors $\mathbf{g}^i$, with

$$d\mathbf{x} = dx^i \mathbf{e}_i = d\Theta^i \mathbf{g}_i \quad , \tag{2.10.5}$$

As the material deforms, the covariant base vectors $\mathbf{g}_i$ deform with the body, being "attached" to the body. However, note that the contravariant base vectors $\mathbf{g}^i$ are not as such attached; they have to be re-evaluated at each step of the deformation anew, so as to ensure that the relevant relations, e.g. $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, are always satisfied.

Example 1

Consider a pure shear deformation, where a square deforms into a parallelogram, as illustrated in Fig. 2.10.2. In this scenario, a unit vector $\mathbf{E}_2$ in the "square" gets mapped to a vector $\mathbf{g}_2$ in the parallelogram[1]. The magnitude of $\mathbf{g}_2$ is $1/\sin\alpha$.

[1] This differs from the example worked through in section 1.16; there, the vector $\mathbf{g}_2$ maintained unit magnitude.

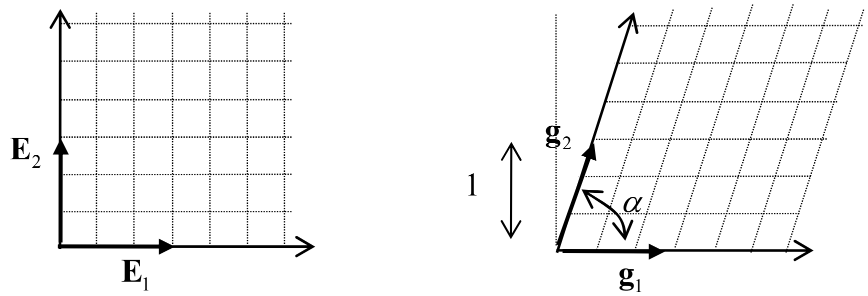


Figure 2.10.2: A pure shear deformation

Consider now a parallelogram (initial condition) deforming into a new parallelogram (the current configuration), as shown in Fig. 2.10.3.

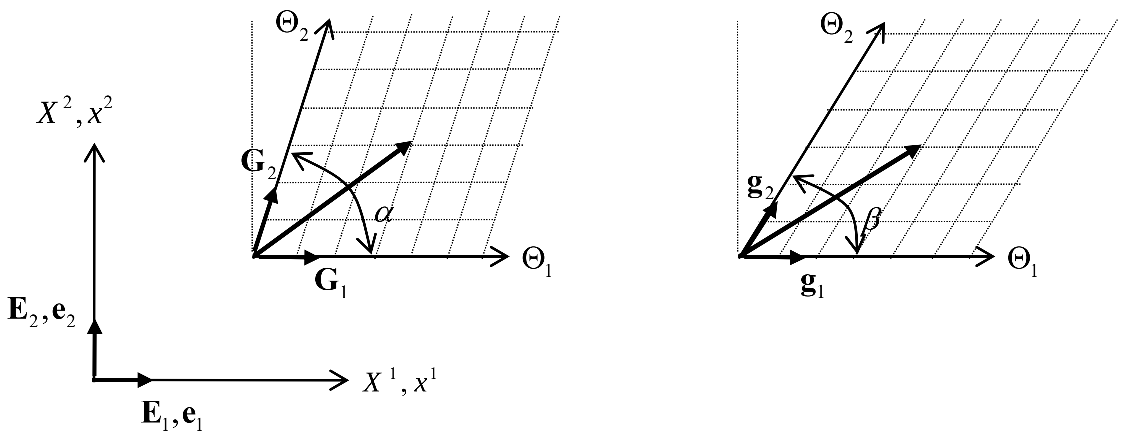


Figure 2.10.3: A pure shear deformation of one parallelogram into another

Keeping in mind that the vector $\mathbf{g}_2$ will be of magnitude $1/\sin\alpha$, the transformation equations 2.10.1 for the configurations shown in Fig. 2.10.3 are[2]

$$\begin{aligned} &\Theta^1 = X^1 - \frac{1}{\tan\alpha}X^2, \quad \Theta^2 = X^2, \quad \Theta^3 = X^3 \\ &X^1 = \Theta^1 + \frac{1}{\tan\alpha}\Theta^2, \quad X^2 = \Theta^2, \quad X^3 = \Theta^3 \\ &\Theta^1 = x^1 - \frac{1}{\tan\beta}x^2, \quad \Theta^2 = x^2, \quad \Theta^3 = x^3 \\ &x^1 = \Theta^1 + \frac{1}{\tan\beta}\Theta^2, \quad x^2 = \Theta^2, \quad x^3 = \Theta^3 \end{aligned} \tag{2.10.6}$$

[2] Constants have been omitted from these expressions (which represent the translation of the "parallelogram origin" from the Cartesian origin).

Following on from §1.16, Eqns. 1.16.19, the covariant base vectors are:

$$\begin{aligned} \mathbf{G}_i &= \frac{\partial X^m}{\partial \Theta^i}\mathbf{E}_m, \qquad \mathbf{G}_1 = \mathbf{E}_1, \quad \mathbf{G}_2 = \frac{1}{\tan\alpha}\mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{G}_3 = \mathbf{E}_3 \\ \mathbf{g}_i &= \frac{\partial X^m}{\partial \Theta^i}\mathbf{e}_m, \qquad \mathbf{g}_1 = \mathbf{e}_1, \quad \mathbf{g}_2 = \frac{1}{\tan\beta}\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{g}_3 = \mathbf{e}_3 \end{aligned} \tag{2.10.7}$$

and the inverse expressions

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{G}_1, \qquad \mathbf{E}_2 = -\frac{1}{\tan\alpha}\mathbf{G}_1 + \mathbf{G}_2, \qquad \mathbf{E}_3 = \mathbf{G}_3 \\ \mathbf{e}_1 &= \mathbf{g}_1, \qquad \mathbf{e}_2 = -\frac{1}{\tan\beta}\mathbf{g}_1 + \mathbf{g}_2, \qquad \mathbf{e}_3 = \mathbf{g}_3 \end{aligned} \tag{2.10.8}$$

Line elements in the configurations can now be expressed as

$$\begin{aligned} d\mathbf{X} &= dX^i\mathbf{E}_i = \frac{d\mathbf{X}}{\partial \Theta^i}d\Theta^i = d\Theta^i\mathbf{G}_i \\ d\mathbf{x} &= dx^i\mathbf{e}_i = \frac{d\mathbf{x}}{\partial \Theta^i}d\Theta^i = d\Theta^i\mathbf{g}_i \end{aligned} \tag{2.10.9}$$

The scale factors, i.e. the magnitudes of the covariant base vectors, are (see Eqns. 1.16.36)

$$\begin{aligned} H_1 &= |\mathbf{G}_1| = 1, \quad H_2 = |\mathbf{G}_2| = \frac{1}{\sin\alpha} \\ h_1 &= |\mathbf{g}_1| = 1, \quad h_2 = |\mathbf{g}_2| = \frac{1}{\sin\beta} \end{aligned} \tag{2.10.10}$$

The contravariant base vectors are (see Eqn. 1.16.23)

$$\begin{aligned} \mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^m}\mathbf{E}_m, \qquad \mathbf{G}^1 = \mathbf{E}_1 - \frac{1}{\tan\alpha}\mathbf{E}_2, \quad \mathbf{G}^2 = \mathbf{E}_2, \quad \mathbf{G}^3 = \mathbf{E}_3 \\ \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial X^m}\mathbf{e}_m, \qquad \mathbf{g}^1 = \mathbf{e}_1 - \frac{1}{\tan\beta}\mathbf{e}_2, \quad \mathbf{g}^2 = \mathbf{e}_2, \quad \mathbf{g}^3 = \mathbf{e}_3 \end{aligned} \tag{2.10.11}$$

and the inverse expressions

$$\mathbf{E}_1 = \mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2, \qquad \mathbf{E}_2 = \mathbf{G}^2, \qquad \mathbf{E}_3 = \mathbf{G}^3$$
$$\mathbf{e}_1 = \mathbf{g}^1 + \frac{1}{\tan\beta}\mathbf{g}^2, \qquad \mathbf{e}_2 = \mathbf{g}^2, \qquad \mathbf{e}_3 = \mathbf{g}^3 \qquad (2.10.12)$$

The magnitudes of the contravariant base vectors, are

$$H^1 = \left|\mathbf{G}^1\right| = \frac{1}{\sin\alpha}, \quad H^2 = \left|\mathbf{G}^2\right| = 1$$
$$h^1 = \left|\mathbf{g}^1\right| = \frac{1}{\sin\beta}, \quad h^2 = \left|\mathbf{g}^2\right| = 1 \qquad (2.10.13)$$

The metric coefficients are (see Eqns. 1.16.27)

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j = \begin{bmatrix} 1 & \frac{1}{\tan\alpha} & 0 \\ \frac{1}{\tan\alpha} & \frac{1}{\sin^2\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G^{ij} = \mathbf{G}^i \cdot \mathbf{G}^j = \begin{bmatrix} \frac{1}{\sin^2\alpha} & -\frac{1}{\tan\alpha} & 0 \\ -\frac{1}{\tan\alpha} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{bmatrix} 1 & \frac{1}{\tan\beta} & 0 \\ \frac{1}{\tan\beta} & \frac{1}{\sin^2\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \begin{bmatrix} \frac{1}{\sin^2\beta} & -\frac{1}{\tan\beta} & 0 \\ -\frac{1}{\tan\beta} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.10.14)$$

The transformation determinants are (consistent with zero volume change), from Eqns. 1.16.32-34,

$$G = \det\left[G_{ij}\right] = \frac{1}{\det\left[G^{ij}\right]} = \left(\det\left[\frac{\partial X^i}{\partial \Theta^j}\right]\right)^2 = J_G^2 = 1$$
$$g = \det\left[g_{ij}\right] = \frac{1}{\det\left[g^{ij}\right]} = \left(\det\left[\frac{\partial x^i}{\partial \Theta^j}\right]\right)^2 = J_g^2 = 1 \qquad (2.10.15)$$

■

Example 2

Consider a motion whereby a cube of material, with sides of length L_0, is transformed into a cylinder of radius R and height H, Fig. 2.10.4.

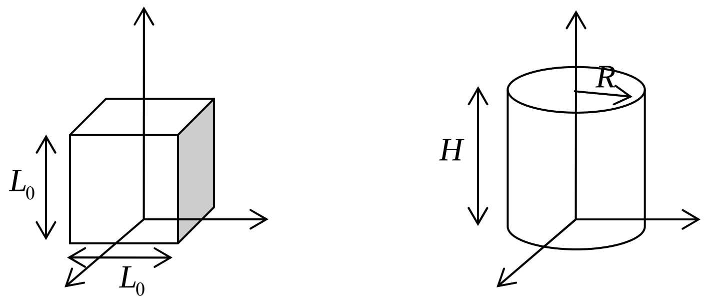


Figure 2.10.4: a cube deformed into a cylinder

A plane view of one quarter of the cube and cylinder are shown in Fig. 2.10.5.

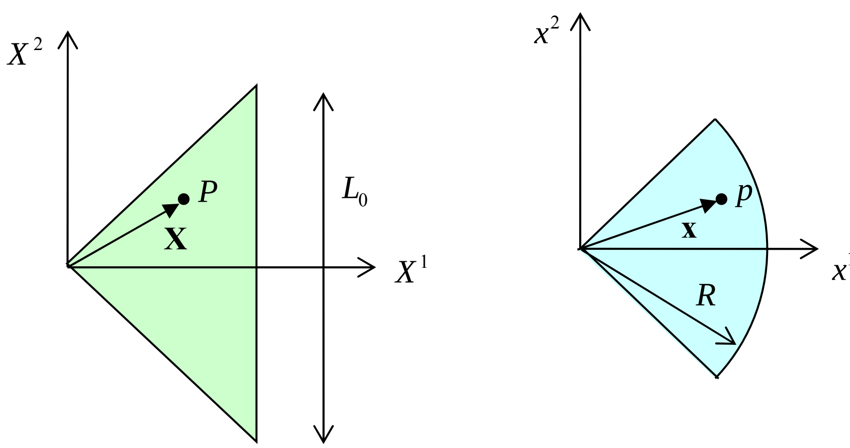


Figure 2.10.5: a cube deformed into a cylinder

The motion and inverse motion are given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \qquad \begin{aligned} x^1 &= \frac{2R}{L_0} \frac{\left(X^1\right)^2}{\sqrt{\left(X^1\right)^2 + \left(X^2\right)^2}} \\ x^2 &= \frac{2R}{L_0} \frac{X^1 X^2}{\sqrt{\left(X^1\right)^2 + \left(X^2\right)^2}} \\ x^3 &= \frac{H}{L_0} X^3 \end{aligned} \qquad (\text{basis: } \mathbf{e}_i) \qquad (2.10.16)$$

and

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}), \qquad \begin{aligned} X^1 &= \frac{L_0}{2R}\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} \\ X^2 &= \frac{L_0}{2R}\frac{x^2}{x^1}\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} \\ X^3 &= \frac{L_0}{H}x^3 \end{aligned} \qquad (\text{basis: } \mathbf{E}_i) \tag{2.10.17}$$

Introducing a set of convected coordinates, Fig. 2.10.6, the material and spatial coordinates are

$$\mathbf{X} = \mathbf{X}(\Theta^1, \Theta^2, \Theta^3), \qquad \begin{aligned} X^1 &= \left(\frac{L_0}{2R}\right)\Theta^1 \\ X^2 &= \left(\frac{L_0}{2R}\right)\Theta^1 \tan\Theta^2 \\ X^3 &= \frac{L_0}{H}\Theta^3 \end{aligned} \tag{2.10.18}$$

and (these are simply cylindrical coordinates)

$$\mathbf{x} = \mathbf{x}(\Theta^1, \Theta^2, \Theta^3), \qquad \begin{aligned} x^1 &= \Theta^1 \cos\Theta^2 \\ x^2 &= \Theta^1 \sin\Theta^2 \\ x^3 &= \Theta^3 \end{aligned} \tag{2.10.19}$$

A typical material particle (denoted by *p*) is shown in Fig. 2.10.6. Note that the position vectors for *p* have the same Θ^i values, since they represent the same material particle.

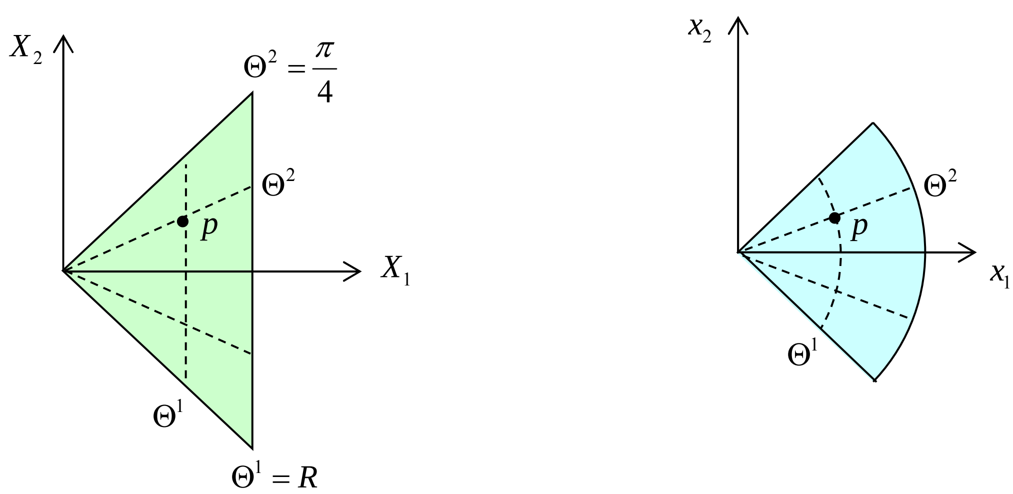


Figure 2.10.6: curvilinear coordinate curves

■

2.10.2 The Deformation Gradient

With convected curvilinear coordinates, the deformation gradient is

$$\begin{aligned}\mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i \\ &= \mathbf{g}_1 \otimes \mathbf{G}^1 + \mathbf{g}_2 \otimes \mathbf{G}^2 + \mathbf{g}_3 \otimes \mathbf{G}^3 , \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\mathbf{g}_i \otimes \mathbf{G}^j\right)\end{aligned} \tag{2.10.20}$$

The deformation gradient operates on a material vector (with contravariant components) $\mathbf{V} = V^i \mathbf{G}_i$, resulting in a spatial tensor $\mathbf{v} = v^i \mathbf{g}_i$ (with the same components $V = v^i$), for example,

$$\mathbf{F} d\mathbf{X} = \left(\mathbf{g}_i \otimes \mathbf{G}^i\right) d\Theta^j \mathbf{G}_j = d\Theta^i \mathbf{g}_i = d\mathbf{x} \tag{2.10.21}$$

To emphasise the point, line elements mapped between the configurations have the same coordinates Θ^i: a line element $d\Theta^1 \mathbf{G}_1 + d\Theta^2 \mathbf{G}_2 + d\Theta^3 \mathbf{G}_3$ gets mapped to

$$\left(\mathbf{g}_1 \otimes \mathbf{G}^1 + \mathbf{g}_2 \otimes \mathbf{G}^2 + \mathbf{g}_3 \otimes \mathbf{G}^3\right)\left(d\Theta^1 \mathbf{G}_1 + d\Theta^2 \mathbf{G}_2 + d\Theta^3 \mathbf{G}_3\right) = d\Theta^1 \mathbf{g}_1 + d\Theta^2 \mathbf{g}_2 + d\Theta^3 \mathbf{g}_3 \tag{2.10.22}$$

This shows also that line elements tangent to the coordinate curves are mapped to new elements tangent to the new coordinate curves; the covariant base vectors $\mathbf{G}_i$ are a field of tangent vectors which get mapped to the new field of tangent vectors $\mathbf{g}_i$, as illustrated in Fig. 2.10.7.

Figure 2.10.7: Vectors tangent to coordinate curves

The deformation gradient **F**, the transpose $\mathbf{F}^{\mathrm{T}}$ and the inverses $\mathbf{F}^{-1}, \mathbf{F}^{-\mathrm{T}}$, map the base vectors in one configuration onto the base vectors in the other configuration (that the $\mathbf{F}^{-1}$ and $\mathbf{F}^{-\mathrm{T}}$ in this equation are indeed the inverses of $\mathbf{F}$ and $\mathbf{F}^{\mathrm{T}}$ follows from 1.16.63):

$$\boxed{\begin{aligned} \mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i \\ \mathbf{F}^{-1} &= \mathbf{G}_i \otimes \mathbf{g}^i \\ \mathbf{F}^{-\mathrm{T}} &= \mathbf{g}^i \otimes \mathbf{G}_i \\ \mathbf{F}^{\mathrm{T}} &= \mathbf{G}^i \otimes \mathbf{g}_i \end{aligned}} \quad \rightarrow \quad \boxed{\begin{aligned} \mathbf{F}\mathbf{G}_i &= \mathbf{g}_i \\ \mathbf{F}^{-1}\mathbf{g}_i &= \mathbf{G}_i \\ \mathbf{F}^{-\mathrm{T}}\mathbf{G}^i &= \mathbf{g}^i \\ \mathbf{F}^{\mathrm{T}}\mathbf{g}^i &= \mathbf{G}^i \end{aligned}} \qquad \textbf{Deformation Gradient} \quad (2.10.23)$$

Thus the tensors $\mathbf{F}$ and $\mathbf{F}^{-1}$ map the covariant base vectors into each other, whereas the tensors $\mathbf{F}^{-\mathrm{T}}$ and $\mathbf{F}^{\mathrm{T}}$ map the contravariant base vectors into each other, as illustrated in Fig. 2.10.8.

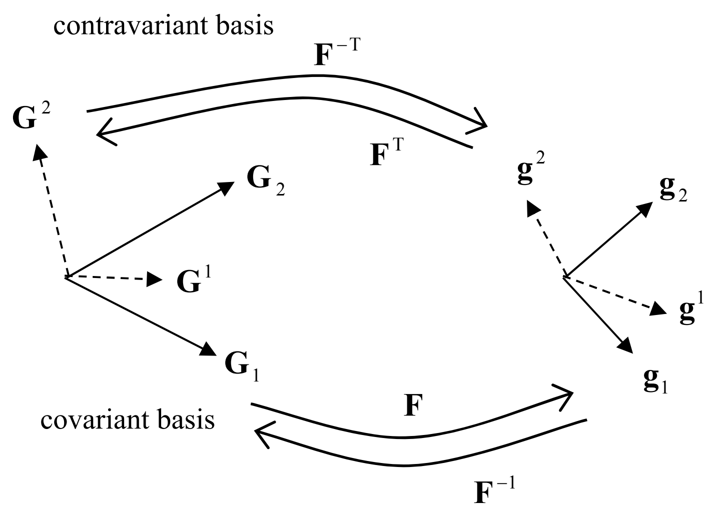


Figure 2.10.8: the deformation gradient, its transpose and the inverses

It was mentioned above how the deformation gradient maps base vectors tangential to the coordinate curves into new vectors tangential to the coordinate curves in the current configuration. In the same way, contravariant base vectors, which are normal to coordinate surfaces, get mapped to normal vectors in the current configuration. For example, the contravariant vector $\mathbf{G}^1$ is normal to the surface of constant Θ^1, and gets mapped through $\mathbf{F}^{-\mathrm{T}}$ to the new vector $\mathbf{g}^1$, which is normal to the surface of constant Θ^1 in the current configuration.

Example 1 continued

Carrying on Example 1 from above, in Cartesian coordinates, 4 corners of an initial parallelogram (see Fig. 2.10.3) get mapped as follows:

$$\begin{aligned}(0,0) &\to (0,0)\\ (1,0) &\to (1,0)\\ (1/\tan\alpha,1) &\to (1/\tan\beta,1)\\ (1+1/\tan\alpha,1) &\to (1+1/\tan\beta,1)\end{aligned} \tag{2.10.24}$$

This corresponds to a deformation gradient with respect to the Cartesian bases:

$$\mathbf{F} = \begin{bmatrix} 1 & \Pi \\ 0 & 1 \end{bmatrix} \quad (\mathbf{E}_i \otimes \mathbf{E}_j),(\mathbf{e}_i \otimes \mathbf{e}_j) \tag{2.10.25}$$

where

$$\Pi = \frac{1}{\tan\beta} - \frac{1}{\tan\alpha} \tag{2.10.26}$$

From the earlier work with example 1, the deformation gradient can be re-expressed in terms of different base vectors:

$$\begin{aligned}\mathbf{F} &= (\mathbf{E}_1 \otimes \mathbf{E}_1) + \Pi(\mathbf{E}_1 \otimes \mathbf{E}_2) + (\mathbf{E}_2 \otimes \mathbf{E}_2)\\ &= (\mathbf{e}_1 \otimes \mathbf{E}_1) + \Pi(\mathbf{e}_1 \otimes \mathbf{E}_2) + (\mathbf{e}_2 \otimes \mathbf{E}_2)\\ &= \mathbf{g}_1 \otimes \left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) + \Pi(\mathbf{g}_1 \otimes \mathbf{G}^2) + \left(-\frac{1}{\tan\beta}\mathbf{g}_1 + \mathbf{g}_2\right) \otimes \mathbf{G}^2\\ &= \mathbf{g}_i \otimes \mathbf{G}^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{g}_i \otimes \mathbf{G}^j)\end{aligned} \tag{2.10.27}$$

which is Eqn. 2.10.20.

In fact, **F** can be expressed in a multitude of different ways, depending on which base vectors are used. For example, from the above, **F** can also be expressed as

$$
\begin{aligned}
\mathbf{F} &= \left(\mathbf{E}_1 \otimes \mathbf{E}_1\right) + \Pi\left(\mathbf{E}_1 \otimes \mathbf{E}_2\right) + \left(\mathbf{E}_2 \otimes \mathbf{E}_2\right) \\
&= \left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) \otimes \left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) + \Pi\left[\left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) \otimes \left(\mathbf{G}^2\right)\right] + \left[\mathbf{G}^2 \otimes \mathbf{G}^2\right] \\
&= \begin{bmatrix} 1 & \frac{1}{\tan\beta} & 0 \\ \frac{1}{\tan\alpha} & \frac{1}{\tan\alpha}\frac{1}{\tan\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\mathbf{G}^i \otimes \mathbf{G}^j\right)
\end{aligned}
\tag{2.10.28}
$$

(This can be verified using Eqn. 2.10.30a below.)

Components of F

The various components of **F** and its inverses and the transposes, with respect to the different bases, are:

$$
\begin{aligned}
\mathbf{F} &= F_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = F^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = F_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = F_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= f_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = f^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = f_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = f_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\
\mathbf{F}^{-1} &= \left(F^{-1}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{-1}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{-1}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{-1}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{-1}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{-1}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{-1}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{-1}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\
\mathbf{F}^{\mathrm{T}} &= \left(F^{\mathrm{T}}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{\mathrm{T}}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{\mathrm{T}}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{\mathrm{T}}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{\mathrm{T}}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{\mathrm{T}}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\
\mathbf{F}^{-\mathrm{T}} &= \left(F^{-\mathrm{T}}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{-\mathrm{T}}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{-\mathrm{T}}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{-\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{-\mathrm{T}}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{-\mathrm{T}}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{-\mathrm{T}}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{-\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j
\end{aligned}
\tag{2.10.29}
$$

The components of **F** with respect to the reference bases $\{\mathbf{G}_i\}, \{\mathbf{G}^i\}$ are

$$\begin{aligned} F_{ij} &= \mathbf{G}_i \mathbf{F} \mathbf{G}_j = \mathbf{G}_i \cdot \mathbf{g}_j = \frac{\partial X^m}{\partial \Theta^i}\frac{\partial x^m}{\partial \Theta^j} \\ F^{ij} &= \mathbf{G}^i \mathbf{F} \mathbf{G}^j = G^{jk} \mathbf{G}^i \cdot \mathbf{g}_k \\ F_i^{\cdot j} &= \mathbf{G}_i \mathbf{F} \mathbf{G}^j = G^{jk} \mathbf{G}_i \cdot \mathbf{g}_k \\ F_{\cdot j}^{i} &= \mathbf{G}^i \mathbf{F} \mathbf{G}_j = \mathbf{G}^i \cdot \mathbf{g}_j = \frac{\partial \Theta^i}{\partial X^m}\frac{\partial x^m}{\partial \Theta^j} \end{aligned} \tag{2.10.30}$$

and similarly for the components with respect to the current bases.

Components of the Base Vectors in different Bases

The base vectors themselves can be expressed alternately:

$$\begin{aligned} \mathbf{g}_i = \mathbf{F}\mathbf{G}_i &= F_{mj}\left(\mathbf{G}^m \otimes \mathbf{G}^j\right)\mathbf{G}_i & &= F_{\cdot j}^{m}\left(\mathbf{G}_m \otimes \mathbf{G}^j\right)\mathbf{G}_i \\ &= F_{mj}\mathbf{G}^m \delta_i^j & &= F_{\cdot j}^{m}\mathbf{G}_m \delta_i^j \\ &= F_{mi}\mathbf{G}^m & &= F_{\cdot i}^{m}\mathbf{G}_m \end{aligned} \tag{2.10.31}$$

showing that some of the components of the deformation gradient can be viewed also as components of the base vectors. Similarly,

$$\mathbf{G}_i = \mathbf{F}^{-1}\mathbf{g}_i = \left(f^{-1}\right)_{mi}\mathbf{g}^m = \left(f^{-1}\right)_{\cdot i}^{m}\mathbf{g}_m \tag{2.10.32}$$

For the contravariant base vectors, one has

$$\begin{aligned} \mathbf{g}^i = \mathbf{F}^{-\mathrm{T}}\mathbf{G}^i &= \left(F^{-\mathrm{T}}\right)^{mj}\left(\mathbf{G}_m \otimes \mathbf{G}_j\right)\mathbf{G}^i & &= \left(F^{-\mathrm{T}}\right)_{m}^{\cdot j}\left(\mathbf{G}^m \otimes \mathbf{G}_j\right)\mathbf{G}^i \\ &= \left(F^{-\mathrm{T}}\right)^{mj}\mathbf{G}_m \delta_j^i & &= \left(F^{-\mathrm{T}}\right)_{m}^{\cdot j}\mathbf{G}^m \delta_j^i \\ &= \left(F^{-\mathrm{T}}\right)^{mi}\mathbf{G}_m & &= \left(F^{-\mathrm{T}}\right)_{m}^{\cdot i}\mathbf{G}^m \end{aligned} \tag{2.10.33}$$

and

$$\mathbf{G}^i = \mathbf{F}^{\mathrm{T}}\mathbf{g}^i = \left(f^{\mathrm{T}}\right)^{mi}\mathbf{g}_m = \left(f^{\mathrm{T}}\right)_{m}^{\cdot i}\mathbf{g}^m \tag{2.10.34}$$

2.10.3 Reduction to Material and Spatial Coordinates

Material Coordinates

Suppose that the material coordinates X^i with Cartesian basis are used (rather than the convected coordinates with curvilinear basis $\mathbf{G}_i$), Fig. 2.10.9. Then

$$\Theta^i \to X^i, \qquad \begin{aligned} \mathbf{G}_i &= \frac{\partial X^j}{\partial \Theta^i}\mathbf{E}_j = \frac{\partial X^j}{\partial X^i}\mathbf{E}_j = \mathbf{E}_i \\ \mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^j}\mathbf{E}^j = \frac{\partial X^i}{\partial X^j}\mathbf{E}^j = \mathbf{E}^i \end{aligned}, \qquad \begin{aligned} \mathbf{g}_i &= \frac{\partial x^j}{\partial \Theta^i}\mathbf{e}_j = \frac{\partial x^j}{\partial X^i}\mathbf{e}_j \\ \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial x^j}\mathbf{e}^j = \frac{\partial X^i}{\partial x^j}\mathbf{e}^j \end{aligned} \tag{2.10.35}$$

and

$$\begin{aligned} \mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{g}_i \otimes \mathbf{E}^i = \frac{\partial x^j}{\partial X^i}\mathbf{e}_j \otimes \mathbf{E}^i = \mathrm{Grad}\mathbf{x} \\ \mathbf{F}^{-1} &= \mathbf{G}_i \otimes \mathbf{g}^i = \mathbf{E}_i \otimes \mathbf{g}^i = \frac{\partial X^i}{\partial x^j}\mathbf{E}_i \otimes \mathbf{e}^j = \mathrm{grad}\mathbf{X} \end{aligned} \tag{2.10.36}$$

which are Eqns. 2.2.2, 2.2.4. Thus $\mathrm{Grad}\mathbf{x}$ is the notation for $\mathbf{F}$ and $\mathrm{grad}\mathbf{X}$ is the notation for $\mathbf{F}^{-1}$, to be used when the material coordinates X_i are used to describe the deformation.

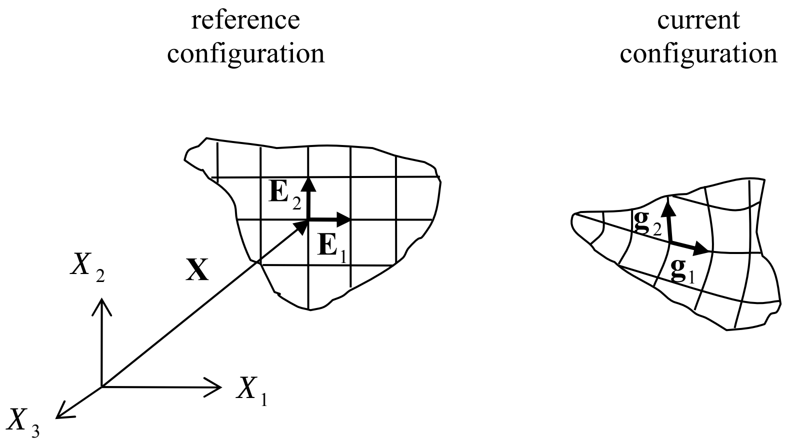


Figure 2.10.9: Material coordinates and deformed basis

Spatial Coordinates

Similarly, when the spatial coordinates x^i are to be used as independent variables, then

$$\Theta^i \to x^i, \qquad \begin{aligned} \mathbf{G}_i &= \frac{\partial X^j}{\partial \Theta^i}\mathbf{E}_j = \frac{\partial X^j}{\partial x^i}\mathbf{E}_j \\ \mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^j}\mathbf{E}^j = \frac{\partial x^i}{\partial X^j}\mathbf{E}^j \end{aligned}, \qquad \begin{aligned} \mathbf{g}_i &= \frac{\partial x^j}{\partial \Theta^i}\mathbf{e}_j = \frac{\partial x^j}{\partial x^i}\mathbf{e}_j = \mathbf{e}_i \\ \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial x^j}\mathbf{e}^j = \frac{\partial x^i}{\partial x^j}\mathbf{e}^j = \mathbf{e}^i \end{aligned} \tag{2.10.37}$$

and

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{e}_i \otimes \mathbf{G}^i = \frac{\partial x^i}{\partial X^j}\mathbf{e}_i \otimes \mathbf{E}^j = \text{Grad}\mathbf{x}$$
$$\mathbf{F}^{-1} = \mathbf{G}_i \otimes \mathbf{g}^i = \mathbf{G}_i \otimes \mathbf{e}^i = \frac{\partial X^j}{\partial x^i}\mathbf{E}_j \otimes \mathbf{e}^i = \text{grad}\mathbf{X} \tag{2.10.38}$$

The descriptions are illustrated in Fig. 2.10.10. Note that the base vectors $\mathbf{G}_i$, $\mathbf{g}_i$ are not the same in each of these cases (curvilinear, material and spatial).

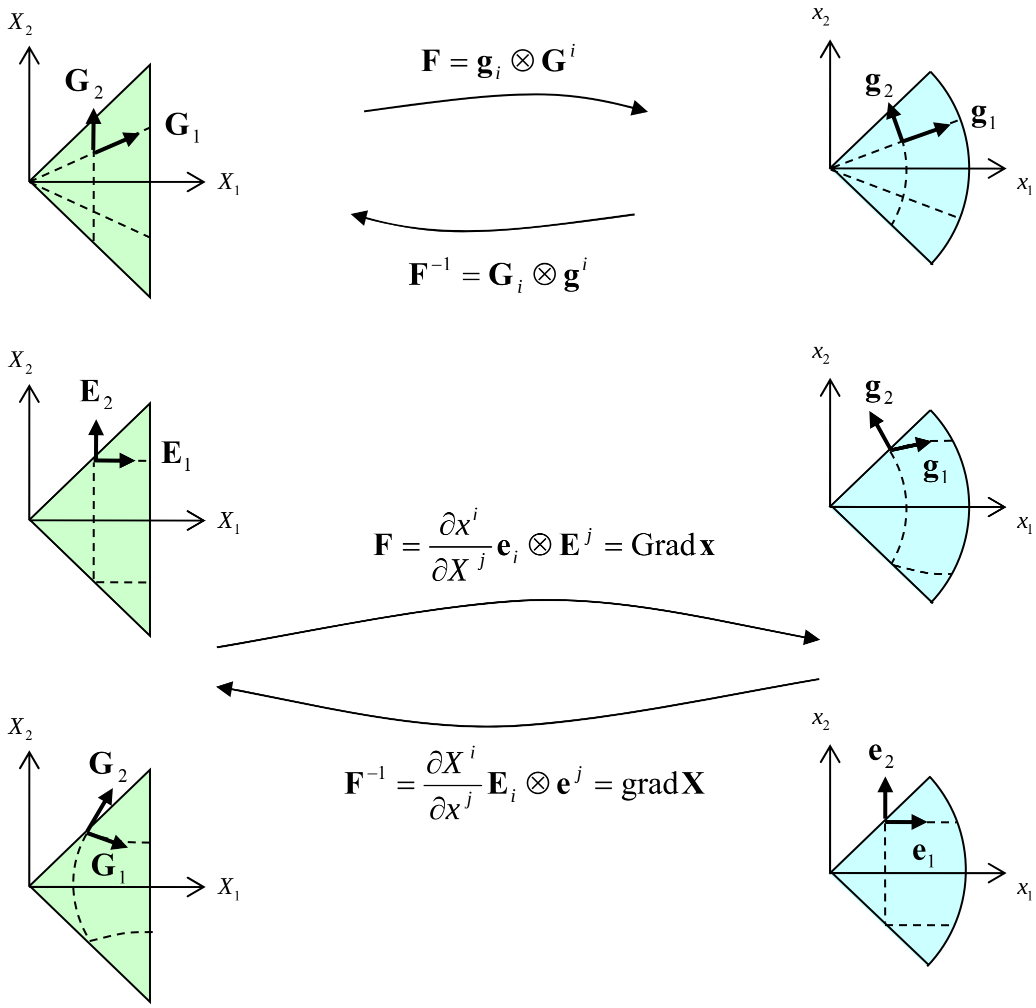


Figure 2.10.10: deformation described using different independent variables

2.10.4 Strain Tensors

The Cauchy-Green tensors

The right Cauchy-Green tensor **C** and the left Cauchy-Green tensor **b** are defined by Eqns. 2.2.10, 2.2.13,

$$\begin{aligned}
\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} &= \left(\mathbf{G}^i \otimes \mathbf{g}_i\right)\left(\mathbf{g}_j \otimes \mathbf{G}^j\right) = g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \equiv C_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \\
\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} &= \left(\mathbf{G}_i \otimes \mathbf{g}^i\right)\left(\mathbf{g}^j \otimes \mathbf{G}_j\right) = g^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \equiv \left(C^{-1}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\
\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} &= \left(\mathbf{g}_i \otimes \mathbf{G}^i\right)\left(\mathbf{G}^j \otimes \mathbf{g}_j\right) = G^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \equiv b^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \\
\mathbf{b}^{-1} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} &= \left(\mathbf{g}^i \otimes \mathbf{G}_i\right)\left(\mathbf{G}_j \otimes \mathbf{g}^j\right) = G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \equiv \left(b^{-1}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned} \qquad (2.10.39)$$

Thus the covariant components of the right Cauchy-Green tensor are the metric coefficients g_{ij}. This highlights the importance of **C**: the $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ give a clear measure of the deformation occurring. (It is possible to evaluate other components of **C**, e.g. C^{ij}, and also its components with respect to the current basis, but only the components C_{ij} with respect to the reference basis are (normally) used in the analysis.)

The Stretch

Now, analogous to 2.2.9, 2.2.12,

$$\begin{aligned}
ds^2 &= d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X}\mathbf{C}d\mathbf{X} \\
dS^2 &= d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x}\mathbf{b}^{-1}d\mathbf{x}
\end{aligned} \qquad (2.10.40)$$

so that the stretches are, analogous to 2.2.17,

$$\begin{aligned}
\lambda^2 &= \frac{ds^2}{dS^2} = \frac{d\mathbf{X}}{|d\mathbf{X}|}\mathbf{C}\frac{d\mathbf{X}}{|d\mathbf{X}|} = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}} \quad \rightarrow d\hat{X}^i C_{ij} d\hat{X}^j \\
\frac{1}{\lambda^2} &= \frac{dS^2}{ds^2} = \frac{d\mathbf{x}}{|d\mathbf{x}|}\mathbf{b}^{-1}\frac{d\mathbf{x}}{|d\mathbf{x}|} = d\hat{\mathbf{x}}\mathbf{b}^{-1}d\hat{\mathbf{x}} \quad \rightarrow d\hat{x}^i \left(b^{-1}\right)_{ij} d\hat{x}^j
\end{aligned} \qquad (2.10.41)$$

The Green-Lagrange and Euler-Almansi Tensors

The Green-Lagrange strain tensor **E** and the Euler-Almansi strain tensor **e** are defined through 2.2.22, 2.2.24,

$$\frac{ds^2 - dS^2}{2} = d\mathbf{X}\frac{1}{2}(\mathbf{C}-\mathbf{I})d\mathbf{X} \equiv d\mathbf{X}\mathbf{E}d\mathbf{X}$$
$$\frac{ds^2 - dS^2}{2} = d\mathbf{x}\frac{1}{2}\left(\mathbf{I}-\mathbf{b}^{-1}\right)d\mathbf{x} \equiv d\mathbf{x}\mathbf{e}d\mathbf{x} \qquad (2.10.42)$$

The components of **E** and **e** can be evaluated through (writing $\mathbf{G} \equiv \mathbf{I}$, the identity tensor expressed in terms of the base vectors in the reference configuration, and $\mathbf{g} \equiv \mathbf{I}$, the identity tensor expressed in terms of the base vectors in the current configuration)

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{G}) = \frac{1}{2}\left(g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j - G_{ij}\mathbf{G}^i \otimes \mathbf{G}^j\right) = \frac{1}{2}\left(g_{ij} - G_{ij}\right)\mathbf{G}^i \otimes \mathbf{G}^j \equiv E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j$$
$$\mathbf{e} = \frac{1}{2}\left(\mathbf{g}-\mathbf{b}^{-1}\right) = \frac{1}{2}\left(g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j - G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right) = \frac{1}{2}\left(g_{ij} - G_{ij}\right)\mathbf{g}^i \otimes \mathbf{g}^j \equiv e_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$$
$$(2.10.43)$$

Note that the components of **E** and **e** with respect to their bases are equal, $E_{ij} = e_{ij}$ (although this is not true regarding their other components, e.g. $E^{ij} \neq e^{ij}$).

Example 1 continued

Carrying on Example 1 from above, consider now an example vector

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad (\mathbf{E}_i) \qquad (2.10.44)$$

The contravariant and covariant components are

$$\mathbf{V} = \begin{bmatrix} V_x - \dfrac{1}{\tan\alpha}V_y \\ V_y \end{bmatrix} \quad (\mathbf{G}_i), \qquad \mathbf{V} = \begin{bmatrix} V_x \\ \dfrac{1}{\tan\alpha}V_x + V_y \end{bmatrix} \quad (\mathbf{G}^i) \qquad (2.10.45)$$

The magnitude of the vector can be calculated through (see Eqn. 1.16.52 and 1.16.49)

$$\begin{aligned} |\mathbf{V}| &= \sqrt{\overset{\mathbf{E}_i}{\mathbf{V}} \cdot \overset{\mathbf{E}_i}{\mathbf{V}}} = \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{\overset{\mathbf{G}_i}{\mathbf{V}} \cdot \overset{\mathbf{G}_i}{\mathbf{V}}} = \sqrt{G_{ij}V^iV^j} = \sqrt{\left(V_x - \frac{V_y}{\tan\alpha}\right)^2 G_{11} + 2\left(V_x - \frac{V_y}{\tan\alpha}\right)V_y G_{12} + V_y^2 G_{22}} \\ &= \sqrt{\overset{\mathbf{G}^i}{\mathbf{V}} \cdot \overset{\mathbf{G}^i}{\mathbf{V}}} = \sqrt{G^{ij}V_iV_j} = \sqrt{V_x^2 G^{11} + 2V_x\left(\frac{V_x}{\tan\alpha} + V_y\right)G^{12} + \left(\frac{V_x}{\tan\alpha} + V_y\right)^2 G^{22}} \end{aligned} \qquad (2.10.46)$$

The new vector is obtained from the deformation gradient:

$$\begin{aligned}\mathbf{v} = \mathbf{F}\overset{\mathbf{E}_i}{\mathbf{V}} &= \begin{bmatrix}1 & \Pi \\ 0 & 1\end{bmatrix}\begin{bmatrix}V_x \\ V_y\end{bmatrix} &&= \begin{bmatrix}V_x + \Pi V_y \\ V_y\end{bmatrix}(\mathbf{e}_i) \\ = \mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} &= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}V_x - \dfrac{V_y}{\tan\alpha} \\ V_y\end{bmatrix} &&= \begin{bmatrix}V_x - \dfrac{1}{\tan\alpha}V_y \\ V_y\end{bmatrix}(\mathbf{g}_i)\end{aligned} \tag{2.10.47}$$

In terms of the contravariant vectors:

$$\mathbf{v} = v_j g^j = \begin{bmatrix}V_x + \Pi V_y \\ \dfrac{1}{\tan\beta}V_x + \left(1 + \dfrac{1}{\tan\beta}\Pi\right)V_y\end{bmatrix}(\mathbf{g}^i) \tag{2.10.48}$$

Note that the contravariant components do not change with the deformation, but the covariant components do in general change with the deformation.

The magnitudes of the vectors before and after deformation are given by the Cauchy-Green strain tensors, whose coefficients are those of the metric tensors (the first of these is the same as 2.10.46)

$$\begin{aligned}\mathbf{V}\cdot\mathbf{V} &= \mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}}\cdot\mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = \overset{\mathbf{g}_i}{\mathbf{v}}\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = \overset{\mathbf{g}_i}{\mathbf{v}}\mathbf{b}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = v^k\mathbf{g}_k G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j v^l \mathbf{g}_l = G_{ij}v^i v^j \\ \mathbf{v}\cdot\mathbf{v} &= \mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}}\cdot\mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} = \overset{\mathbf{G}_i}{\mathbf{V}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} = \overset{\mathbf{G}_i}{\mathbf{V}}\mathbf{C}\overset{\mathbf{G}_i}{\mathbf{V}} = V^k\mathbf{G}_k g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j V^l \mathbf{G}_l = g_{ij}V^i V^j\end{aligned} \tag{2.10.49}$$

From this, the magnitude of the vector after deformation is

$$\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{g_{ij}V^iV^j} = \sqrt{\left(V_x^2 + V_y^2\right) + \Pi V_y\left(2V_x + \Pi V_y\right)} \tag{2.10.50}$$

2.10.5 Intermediate Configurations

Stretch and Rotation Tensors

The polar decompositions $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}$ have been described in §2.2.5. The decompositions are illustrated in Fig. 2.10.11. In the material decomposition, the material is first stretched by $\mathbf{U}$ and then rotated by $\mathbf{R}$. Let the base vectors in the associated intermediate configuration be $\{\hat{\mathbf{g}}_i\}$. Similarly, in the spatial decomposition, the material is first rotated by $\mathbf{R}$ and then stretched by $\mathbf{v}$. Let the base vectors in the associated intermediate configuration in this case be $\{\mathbf{G}_i\}$. Then, analogous to Eqn. 2.10.23, {▲Problem 1}

$$\begin{aligned} \mathbf{U} &= \hat{\mathbf{g}}_i \otimes \mathbf{G}^i \\ \mathbf{U}^{-1} &= \mathbf{G}_i \otimes \hat{\mathbf{g}}^i \\ \mathbf{U}^{-\mathrm{T}} &= \hat{\mathbf{g}}^i \otimes \mathbf{G}_i \\ \mathbf{U}^{\mathrm{T}} &= \mathbf{G}^i \otimes \hat{\mathbf{g}}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{U}\mathbf{G}_i &= \hat{\mathbf{g}}_i \\ \mathbf{U}^{-1}\hat{\mathbf{g}}_i &= \mathbf{G}_i \\ \mathbf{U}^{-\mathrm{T}}\mathbf{G}^i &= \hat{\mathbf{g}}^i \\ \mathbf{U}^{\mathrm{T}}\hat{\mathbf{g}}^i &= \mathbf{G}^i \end{aligned} \qquad (2.10.51)$$

$$\begin{aligned} \mathbf{v} &= \mathbf{g}_i \otimes \hat{\mathbf{G}}^i \\ \mathbf{v}^{-1} &= \hat{\mathbf{G}}_i \otimes \mathbf{g}^i \\ \mathbf{v}^{-\mathrm{T}} &= \mathbf{g}^i \otimes \hat{\mathbf{G}}_i \\ \mathbf{v}^{\mathrm{T}} &= \hat{\mathbf{G}}^i \otimes \mathbf{g}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{v}\hat{\mathbf{G}}_i &= \mathbf{g}_i \\ \mathbf{v}^{-1}\mathbf{g}_i &= \hat{\mathbf{G}}_i \\ \mathbf{v}^{-\mathrm{T}}\hat{\mathbf{G}}^i &= \mathbf{g}^i \\ \mathbf{v}^{\mathrm{T}}\mathbf{g}^i &= \hat{\mathbf{G}}^i \end{aligned} \qquad (2.10.52)$$

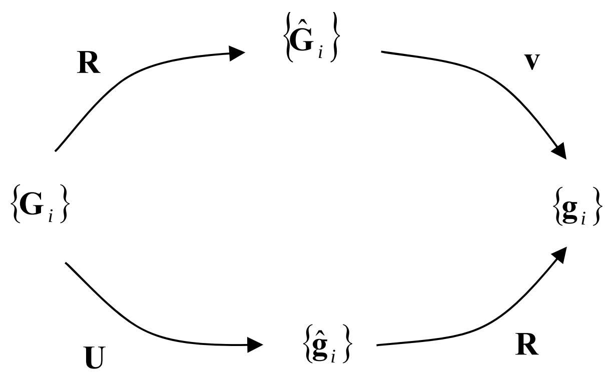


Figure 2.10.11: the material and spatial polar decompositions

Note that **U** and **v** symmetric, $\mathbf{U} = \mathbf{U}^{\mathrm{T}}$, $\mathbf{v} = \mathbf{v}^{\mathrm{T}}$, so

$$\begin{aligned} \mathbf{U} &= \hat{\mathbf{g}}_i \otimes \mathbf{G}^i = \mathbf{G}^i \otimes \hat{\mathbf{g}}_i \\ \mathbf{U}^{-1} &= \mathbf{G}_i \otimes \hat{\mathbf{g}}^i = \hat{\mathbf{g}}^i \otimes \mathbf{G}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{U}\mathbf{G}_i &= \hat{\mathbf{g}}_i, \quad \mathbf{U}\hat{\mathbf{g}}^i = \mathbf{G}^i \\ \mathbf{U}^{-1}\hat{\mathbf{g}}_i &= \mathbf{G}_i, \quad \mathbf{U}^{-1}\mathbf{G}^i = \hat{\mathbf{g}}^i \end{aligned} \qquad (2.10.53)$$

$$\begin{aligned} \mathbf{v} &= \mathbf{g}_i \otimes \hat{\mathbf{G}}^i = \hat{\mathbf{G}}^i \otimes \mathbf{g}_i \\ \mathbf{v}^{-1} &= \hat{\mathbf{G}}_i \otimes \mathbf{g}^i = \mathbf{g}^i \otimes \hat{\mathbf{G}}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{v}\hat{\mathbf{G}}_i &= \mathbf{g}_i, \quad \mathbf{v}\mathbf{g}^i = \hat{\mathbf{G}}^i \\ \mathbf{v}^{-1}\mathbf{g}_i &= \hat{\mathbf{G}}_i, \quad \mathbf{v}^{-1}\hat{\mathbf{G}}^i = \mathbf{g}^i \end{aligned} \qquad (2.10.54)$$

Similarly, for the rotation tensor, with **R** orthogonal, $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$,

$$\begin{aligned} \mathbf{R} &= \hat{\mathbf{G}}_i \otimes \mathbf{G}^i = \hat{\mathbf{G}}^i \otimes \mathbf{G}_i \\ \mathbf{R}^{\mathrm{T}} &= \mathbf{G}_i \otimes \hat{\mathbf{G}}^i = \mathbf{G}^i \otimes \hat{\mathbf{G}}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{R}\mathbf{G}_i &= \hat{\mathbf{G}}_i, \quad \mathbf{R}\mathbf{G}^i = \hat{\mathbf{G}}^i \\ \mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}_i &= \mathbf{G}_i, \quad \mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}^i = \mathbf{G}^i \end{aligned} \qquad (2.10.55)$$

$$\begin{aligned} \mathbf{R} &= \mathbf{g}_i \otimes \hat{\mathbf{g}}^i = \mathbf{g}^i \otimes \hat{\mathbf{g}}_i \\ \mathbf{R}^{\mathrm{T}} &= \hat{\mathbf{g}}_i \otimes \mathbf{g}^i = \hat{\mathbf{g}}^i \otimes \mathbf{g}_i \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathbf{R}\hat{\mathbf{g}}_i &= \mathbf{g}_i, \quad \mathbf{R}\hat{\mathbf{g}}^i = \mathbf{g}^i \\ \mathbf{R}^{\mathrm{T}}\mathbf{g}_i &= \hat{\mathbf{g}}_i, \quad \mathbf{R}^{\mathrm{T}}\mathbf{g}^i = \hat{\mathbf{g}}^i \end{aligned} \qquad (2.10.56)$$

The above relations can be checked using Eqns. 2.10.23 and $\mathbf{F} = \mathbf{RU}$, $\mathbf{F} = \mathbf{vR}$, $\mathbf{v}^{-1} = \mathbf{RF}^{-1}$, etc.

Various relations between the base vectors can be derived, for example,

$$\begin{aligned}
\hat{\mathbf{G}}_i \cdot \mathbf{g}_j &= (\mathbf{R}\mathbf{G}_i)\cdot(\mathbf{R}\hat{\mathbf{g}}_j) = \mathbf{G}_i \mathbf{R}^{\mathrm{T}}\mathbf{R}\hat{\mathbf{g}}_j &&= \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\hat{\mathbf{G}}^i \cdot \mathbf{g}^j &= \qquad \cdots &&= \mathbf{G}^i \cdot \hat{\mathbf{g}}^j \\
\hat{\mathbf{G}}^i \cdot \mathbf{g}_j &= \qquad \cdots &&= \mathbf{G}^i \cdot \hat{\mathbf{g}}_j \\
\hat{\mathbf{G}}_i \cdot \mathbf{g}^j &= \qquad \cdots &&= \mathbf{G}_i \cdot \hat{\mathbf{g}}^j
\end{aligned} \tag{2.10.57}$$

Deformation Gradient Relationship between Bases

The various base vectors are related above through the stretch and rotation tensors. The intermediate bases are related directly through the deformation gradient. For example, from 2.10.53a, 2.10.55b,

$$\hat{\mathbf{g}}_i = \mathbf{U}\mathbf{G}_i = \mathbf{U}\mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}_i = \mathbf{F}^{\mathrm{T}}\hat{\mathbf{G}}_i \tag{2.10.58}$$

In the same way,

$$\begin{aligned}
\hat{\mathbf{g}}_i &= \mathbf{F}^{\mathrm{T}}\hat{\mathbf{G}}_i \\
\hat{\mathbf{g}}^i &= \mathbf{F}^{-1}\hat{\mathbf{G}}^i \\
\hat{\mathbf{G}}_i &= \mathbf{F}^{-\mathrm{T}}\hat{\mathbf{g}}_i \\
\hat{\mathbf{G}}^i &= \mathbf{F}\hat{\mathbf{g}}^i
\end{aligned} \tag{2.10.59}$$

Tensor Components

The stretch and rotation tensors can be decomposed along any of the bases. For $\mathbf{U}$ the most natural bases would be $\{\mathbf{G}_i\}$ and $\{\mathbf{G}^i\}$, for example,

$$\begin{aligned}
\mathbf{U} &= U_{ij}\mathbf{G}^i \otimes \mathbf{G}^j, \quad & U_{ij} &= \mathbf{G}_i\mathbf{U}\mathbf{G}_j = \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U} &= U^{ij}\mathbf{G}_i \otimes \mathbf{G}_j, \quad & U^{ij} &= \mathbf{G}^i\mathbf{U}\mathbf{G}^j = G^{im}\mathbf{G}^j \cdot \hat{\mathbf{g}}_m \\
\mathbf{U} &= U^{i}_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j, \quad & U^{i}_{\cdot j} &= \mathbf{G}^i\mathbf{U}\mathbf{G}_j = \mathbf{G}^i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U} &= U_{i}^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j, \quad & U_{i}^{\cdot j} &= \mathbf{G}_i\mathbf{U}\mathbf{G}^j = \hat{\mathbf{g}}_i \cdot \mathbf{G}^j
\end{aligned} \tag{2.10.60}$$

with $U_{ij} = U_{ji}, U^{ij} = U^{ji}, U^{i}_{\cdot j} = U_{j}^{\cdot i}, U_{i}^{\cdot j} = U^{j}_{\cdot i}$. One also has

$$\begin{aligned}
\mathbf{v} &= v_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j, \quad v_{ij} = \hat{\mathbf{G}}_i \mathbf{v} \hat{\mathbf{G}}_j = \hat{\mathbf{G}}_i \cdot \mathbf{g}_j \\
\mathbf{v} &= v^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j, \quad v^{ij} = \hat{\mathbf{G}}^i \mathbf{v} \hat{\mathbf{G}}^j = \hat{G}^{im}\hat{\mathbf{G}}^j \cdot \mathbf{g}_m \\
\mathbf{v} &= v^i_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j, \quad v^i_{\cdot j} = \hat{\mathbf{G}}^i \mathbf{v} \hat{\mathbf{G}}_j = \hat{\mathbf{G}}^i \cdot \mathbf{g}_j \\
\mathbf{v} &= v_i^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j, \quad v_i^{\cdot j} = \hat{\mathbf{G}}_i \mathbf{v} \hat{\mathbf{G}}^j = \mathbf{g}_i \cdot \hat{\mathbf{G}}^j
\end{aligned} \tag{2.10.61}$$

with similar symmetry. Also,

$$\begin{aligned}
\mathbf{U}^{-1} &= \left(U^{-1}\right)_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j, \quad \left(U^{-1}\right)_{ij} = \hat{\mathbf{g}}_i \mathbf{U}^{-1} \hat{\mathbf{g}}_j = \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j, \quad \left(U^{-1}\right)^{ij} = \hat{\mathbf{g}}^i \mathbf{U}^{-1} \hat{\mathbf{g}}^j = \hat{g}^{im}\mathbf{G}_m \cdot \hat{\mathbf{g}}^j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)^i_{\cdot j}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j, \quad \left(U^{-1}\right)^i_{\cdot j} = \hat{\mathbf{g}}^i \mathbf{U}^{-1} \hat{\mathbf{g}}_j = \hat{\mathbf{g}}^i \cdot \mathbf{G}_j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)_i^{\cdot j}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j, \quad \left(U^{-1}\right)_i^{\cdot j} = \hat{\mathbf{g}}_i \mathbf{U}^{-1} \hat{\mathbf{g}}^j = \mathbf{G}_i \cdot \hat{\mathbf{g}}^j
\end{aligned} \tag{2.10.62}$$

and

$$\begin{aligned}
\mathbf{v}^{-1} &= \left(v^{-1}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \quad \left(v^{-1}\right)_{ij} = \mathbf{g}_i \mathbf{v}^{-1} \mathbf{g}_j = \hat{\mathbf{G}}_i \cdot \mathbf{g}_j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j, \quad \left(v^{-1}\right)^{ij} = \mathbf{g}^i \mathbf{v}^{-1} \mathbf{g}^j = g^{mj}\hat{\mathbf{G}}_m \cdot \mathbf{g}^i \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j, \quad \left(v^{-1}\right)^i_{\cdot j} = \mathbf{g}^i \mathbf{v}^{-1} \mathbf{g}_j = \mathbf{g}^i \cdot \hat{\mathbf{G}}_j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j, \quad \left(v^{-1}\right)_i^{\cdot j} = \mathbf{g}_i \mathbf{v}^{-1} \mathbf{g}^j = \hat{\mathbf{G}}_i \cdot \mathbf{g}^j
\end{aligned} \tag{2.10.63}$$

with similar symmetry. Note that, comparing 2.10.60a, 2.10.61a, 2.10.62a, 2.10.63a and using 2.10.57,

$$\begin{aligned}
\mathbf{U} &= U_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \\
\mathbf{v} &= v_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned} \qquad U_{ij} = \left(U^{-1}\right)_{ij} = v_{ij} = \left(v^{-1}\right)_{ij} \tag{2.10.64}$$

Now note that rotations preserve vectors lengths and, in particular, preserve the metric, i.e.,

$$\begin{aligned}
G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j &= \hat{G}_{ij} = \hat{\mathbf{G}}_i \cdot \hat{\mathbf{G}}_j \\
g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j &= \hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j
\end{aligned} \tag{2.10.65}$$

Thus, again using 2.10.57, and 2.10.60-2.10.63, the contravariant components of the above tensors are also equal, $U^{ij} = \left(U^{-1}\right)^{ij} = v^{ij} = \left(v^{-1}\right)^{ij}$.

As mentioned, the tensors can be decomposed along other bases, for example,

$$\mathbf{v} = v^{ij}\mathbf{g}_i \otimes \mathbf{g}_j, \quad v^{ij} = \mathbf{g}^i \mathbf{v} \mathbf{g}^j = \hat{\mathbf{G}}^i \cdot \mathbf{g}^j \tag{2.10.66}$$

2.10.6 Eigenvectors and Eigenvalues

Analogous to §2.2.5, the eigenvalues of **C** are determined from the eigenvalue problem

$$\det(\mathbf{C} - \lambda_{\mathbf{C}}\mathbf{I}) = 0 \tag{2.10.67}$$

leading to the characteristic equation 1.11.5

$$\lambda_{\mathbf{C}}^3 - \mathrm{I}_{\mathbf{C}}\lambda_{\mathbf{C}}^2 + \mathrm{II}_{\mathbf{C}}\lambda_{\mathbf{C}} - \mathrm{III}_{\mathbf{C}} = 0 \tag{2.10.68}$$

with principal scalar invariants 1.11.6-7

$$\begin{aligned} \mathrm{I}_{\mathbf{C}} &= \mathrm{tr}\mathbf{C} = A_i^i = \lambda_{\mathbf{C}1} + \lambda_{\mathbf{C}2} + \lambda_{\mathbf{C}3} \\ \mathrm{II}_{\mathbf{C}} &= \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right] = \tfrac{1}{2}\left(C_i^i C_j^j - C_j^i C_i^j\right) = \lambda_{\mathbf{C}1}\lambda_{\mathbf{C}2} + \lambda_{\mathbf{C}2}\lambda_{\mathbf{C}3} + \lambda_{\mathbf{C}3}\lambda_{\mathbf{C}1} \\ \mathrm{III}_{\mathbf{C}} &= \det\mathbf{C} = \varepsilon_{ijk} C_1^i C_2^j C_3^k = \lambda_{\mathbf{C}1}\lambda_{\mathbf{C}2}\lambda_{\mathbf{C}3} \end{aligned} \tag{2.10.69}$$

The eigenvectors are the principal material directions $\hat{\mathbf{N}}_i$, with

$$(\mathbf{C} - \lambda_i \mathbf{I})\hat{\mathbf{N}}_i = \mathbf{0} \tag{2.10.70}$$

The spectral decomposition is then

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \tag{2.10.71}$$

where $\lambda_{\mathbf{C}i} = \lambda_i^2$ and the λ_i are the stretches. The remaining spectral decompositions in 2.2.37 hold also. Note also that the rotation tensor in terms of principal directions is (see 2.2.35)

$$\mathbf{R} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}^i = \hat{\mathbf{n}}^i \otimes \hat{\mathbf{N}}_i \tag{2.10.72}$$

where $\hat{\mathbf{n}}_i$ are the spatial principal directions.

2.10.7 Displacement and Displacement Gradients

Consider the displacement **u** of a material particle. This can be written in terms of covariant components U_i and u_i:

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \equiv U_i \mathbf{G}^i = u_i \mathbf{g}^i . \tag{2.10.73}$$

The covariant derivative of **u** can be expressed as

$$\frac{\partial \mathbf{u}}{\partial \Theta^i} = U_m\big|_i \mathbf{G}^m = u_m\big\|_i \mathbf{g}^m \tag{2.10.74}$$

The single line refers to covariant differentiation with respect to the undeformed basis, i.e. the Christoffel symbols to use are functions of the G_{ij}. The double line refers to covariant differentiation with respect to the deformed basis, i.e. the Christoffel symbols to use are functions of the g_{ij}.

Alternatively, the covariant derivative can be expressed as

$$\frac{\partial \mathbf{u}}{\partial \Theta^i} = \frac{\partial \mathbf{x}}{\partial \Theta^i} - \frac{\partial \mathbf{X}}{\partial \Theta^i} = \mathbf{g}_i - \mathbf{G}_i \tag{2.10.75}$$

and so

$$\begin{aligned} \mathbf{g}_i &= \mathbf{G}_i + U_m\big|_i \mathbf{G}^m = \left(\delta_i^m + U^m\big|_i\right)\mathbf{G}_m = F_{\cdot i}^{m}\mathbf{G}_m \\ \mathbf{G}_i &= \mathbf{g}_i - u_m\big\|_i \mathbf{g}^m = \left(\delta_i^m - u^m\big\|_i\right)\mathbf{g}_m = \left(f^{-1}\right)_{\cdot i}^{m}\mathbf{g}_m \end{aligned} \tag{2.10.76}$$

The last equalities following from 2.10.31-32.

The components of the Green-Lagrange and Euler-Almansi strain tensors 2.10.43 can be written in terms of displacements using relations 2.10.76 {▲Problem 2}:

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(U_i\big|_j + U_j\big|_i + U_n\big|_i U^n\big|_j\right) \\ e_{ij} &= \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(u_i\big\|_j + u_j\big\|_i - u_n\big\|_i u^n\big\|_j\right) \end{aligned} \tag{2.10.77}$$

In terms of spatial coordinates, $\Theta^i = X^i$, $\mathbf{G}_i = \mathbf{E}_i$, $\mathbf{g}_i = \left(\partial x^j / \partial X^i\right)\mathbf{e}_j$, $U_i\big|_j = \partial U_i / \partial X^j$, the components of the Euler-Lagrange strain tensor are

$$E_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(\frac{\partial x^m}{\partial X^i}\frac{\partial x^n}{\partial X^j}\delta_{mn} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X^j} + \frac{\partial U_j}{\partial X^i} + \frac{\partial U_k}{\partial X^i}\frac{\partial U_k}{\partial X^j}\right) \qquad (2.10.78)$$

which is 2.2.46.

2.10.8 The Deformation of Area and Volume Elements

Differential Volume Element

Consider a differential volume element formed by the elements $d\Theta^i \mathbf{G}_i$ in the undeformed configuration, Eqn. 1.16.43:

$$dV = \sqrt{G}\, d\Theta^1 d\Theta^2 d\Theta^3 \qquad (2.10.79)$$

where, Eqn. 1.16.31,1.16.34,

$$\sqrt{G} = \sqrt{\det\left[G_{ij}\right]}, \quad G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \qquad (2.10.80)$$

The *same* volume element in the deformed configuration is determined by the elements $d\Theta^i \mathbf{g}_i$:

$$dv = \sqrt{g}\, d\Theta^1 d\Theta^2 d\Theta^3 \qquad (2.10.81)$$

where

$$\sqrt{g} = \sqrt{\det\left[g_{ij}\right]}, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \qquad (2.8.82)$$

From 1.16.53 *et seq.*, 2.10.11,

$$\begin{aligned}\sqrt{g} &= \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \\ &= F^i_{\cdot 1} F^j_{\cdot 2} F^k_{\cdot 3} \mathbf{G}_i \cdot \mathbf{G}_j \times \mathbf{G}_k \\ &= F^i_{\cdot 1} F^j_{\cdot 2} F^k_{\cdot 3} \varepsilon_{ijk} \sqrt{G} \\ &= \sqrt{G} \det \mathbf{F}\end{aligned} \qquad (2.10.83)$$

where ε_{ijk} is the Cartesian permutation symbol, and so the Jacobian determinant is (see 2.2.53)

$$J = \frac{dv}{dV} = \frac{\sqrt{g}}{\sqrt{G}} = \det \mathbf{F} \qquad (2.10.84)$$

and $\det \mathbf{F}$ is the determinant of the matrix with components $F_{.j}^{i}$.

Differential Area Element

Consider a differential surface (parallelogram) element in the undeformed configuration, bounded by two vector elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, and with unit normal $\hat{\mathbf{N}}$. Then the vector normal to the surface element and with magnitude equal to the area of the surface is, using 1.16.54, given by

$$\hat{\mathbf{N}}dS = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = d\Theta^{(1)i}\mathbf{G}_i \times d\Theta^{(2)j}\mathbf{G}_j = e_{ijk}^{(\mathbf{G})} d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{G}^k \qquad (2.10.85)$$

where $e_{ijk}^{(\mathbf{G})}$ is the permutation symbol associated with the basis $\mathbf{G}_i$, i.e.

$$e_{ijk}^{(\mathbf{G})} = \varepsilon_{ijk}\mathbf{G}_i \cdot \mathbf{G}_j \times \mathbf{G}_k = \varepsilon_{ijk}\sqrt{G}\,. \qquad (2.10.86)$$

Using $\mathbf{G}^k = \mathbf{F}^{\mathrm{T}}\mathbf{g}^k$, one has

$$\hat{\mathbf{N}}dS = \varepsilon_{ijk}\sqrt{G}\,d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{F}^{\mathrm{T}}\mathbf{g}^k \qquad (2.10.87)$$

Similarly, the surface vector in the deformed configuration with unit normal $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}}ds = d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = d\Theta^{(1)i}\mathbf{g}_i \times d\Theta^{(2)j}\mathbf{g}_j = e_{ijk}^{(\mathbf{g})} d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{g}^k \qquad (2.10.88)$$

where $e_{ijk}^{(\mathbf{g})}$ is the permutation symbol associated with the basis $\mathbf{g}_i$, i.e.

$$e_{ijk}^{(\mathbf{g})} = \varepsilon_{ijk}\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \varepsilon_{ijk}\sqrt{g}\,. \qquad (2.10.89)$$

Comparing the two expressions for the areas in the undeformed and deformed configurations, 2.10.87-88, one finds that

$$\hat{\mathbf{n}}ds = \sqrt{\frac{g}{G}}\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS = (\det \mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS \qquad (2.10.90)$$

which is Nanson's relation, Eqn. 2.2.59. This is consistent with was said earlier in relation to Fig. 2.10.8 and the contravariant bases: $\mathbf{F}^{-\mathrm{T}}$ maps vectors normal to the coordinate curves in the initial configuration into corresponding vectors normal to the coordinate curves in the current configuration.

2.10.9 Problems

1. Derive the relations 2.10.51.
2. Use relations 2.10.76, with $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$, to derive 2.10.77

$$E_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(U_i\big|_j + U_j\big|_i + U_n\big|_i U^n\big|_j\right)$$

$$e_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(u_i\big\|_j + u_j\big\|_i - u_n\big\|_i u^n\big\|_j\right)$$

Convected Coordinates: Time Rates of Change

In this section, the time derivatives of kinematic tensors described in §2.4-2.6 are now described using convected coordinates.

2.11.1 Deformation Rates

Time Derivatives of the Base Vectors and the Deformation Gradient

The material time derivatives of the material base vectors are zero: $\dot{\mathbf{G}}_i = \dot{\mathbf{G}}^i = 0$. The material time derivatives of the deformed base vectors are, from 2.10.23, (and using $\dot{\mathbf{I}} = d\left(\mathbf{F}\mathbf{F}^{-1}\right)/dt = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\dot{\mathbf{F}}^{-1}$)

$$\begin{aligned} \dot{\mathbf{g}}_i &= \dot{\mathbf{F}}\mathbf{G}_i = \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{g}_i = -\mathbf{F}\dot{\mathbf{F}}^{-1}\mathbf{g}_i \\ \dot{\mathbf{g}}^i &= \dot{\mathbf{F}}^{-\mathrm{T}}\mathbf{G}^i = \dot{\mathbf{F}}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{g}^i = -\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{g}^i \end{aligned} \qquad (2.11.1)$$

with, again from 2.10.23,

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{g}}_i \otimes \mathbf{G}^i \\ \dot{\mathbf{F}}^{-1} &= \mathbf{G}_i \otimes \dot{\mathbf{g}}^i \\ \dot{\mathbf{F}}^{-\mathrm{T}} &= \dot{\mathbf{g}}^i \otimes \mathbf{G}_i \\ \dot{\mathbf{F}}^{\mathrm{T}} &= \mathbf{G}^i \otimes \dot{\mathbf{g}}_i \end{aligned} \qquad (2.11.2)$$

The Velocity Gradient

The velocity gradient is defined by 2.5.2, $\mathbf{l} = \operatorname{grad}\mathbf{v}$, so that, using 1.16.23,

$$\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial x^i} \otimes \mathbf{e}^i = \frac{\partial \mathbf{v}}{\partial \Theta^j}\frac{\partial \Theta^j}{\partial x^i} \otimes \mathbf{e}^i = \frac{\partial \mathbf{v}}{\partial \Theta^j} \otimes \mathbf{g}^j \qquad (2.11.3)$$

Also, from 1.16.19,

$$\dot{\mathbf{g}}_i = \frac{\partial \dot{\mathbf{x}}}{\partial \Theta^i} = \frac{\partial \mathbf{v}}{\partial \Theta^i} \qquad (2.11.4)$$

so that, as an alternative to 2.11.3,

$$\mathbf{l} = \dot{\mathbf{g}}_i \otimes \mathbf{g}^i \qquad (2.11.5)$$

The components of the spatial velocity gradient are

$$\begin{aligned} l_{ij} &= \mathbf{g}_i \mathbf{l} \mathbf{g}_j = \mathbf{g}_i \cdot \dot{\mathbf{g}}_j \\ l^i_{\cdot j} &= \mathbf{g}^i \mathbf{l} \mathbf{g}_j = \mathbf{g}^i \cdot \dot{\mathbf{g}}_j \\ l_i^{\cdot j} &= \mathbf{g}_i \mathbf{l} \mathbf{g}^j = g^{mj} \mathbf{g}_i \cdot \dot{\mathbf{g}}_m = \mathbf{g}_i \cdot \dot{\mathbf{g}}^j \\ l^{ij} &= \mathbf{g}^i \mathbf{l} \mathbf{g}^j = \mathbf{g}^i \cdot \dot{\mathbf{g}}^j \end{aligned} \qquad (2.11.6)$$

Convected Bases

From 2.11.1, 2.11.2 and 2.11.5,

$$\begin{aligned} \dot{\mathbf{g}}_i &= \mathbf{l}\mathbf{g}_i & \dot{\mathbf{g}}^i &= -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \\ &= \mathbf{g}_i \mathbf{l}^{\mathrm{T}} & &= -\mathbf{g}^i \mathbf{l} \end{aligned} \qquad (2.11.7)$$

Contracting the first of these with $d\Theta^i$ leads to

$$\dot{\mathbf{g}}_i d\Theta^i = \mathbf{l}\mathbf{g}_i d\Theta^i \qquad (2.11.8)$$

which is equivalent to 2.5.1, $d\mathbf{v} = \mathbf{l} d\mathbf{x}$.

Time Derivatives of the Deformation Gradient in terms of the Velocity Gradient

Eqns. 2.11.2 can also be re-expressed using Eqns. 2.11.7:

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{g}}_i \otimes \mathbf{G}^i = \mathbf{g}_i \mathbf{l}^{\mathrm{T}} \otimes \mathbf{G}^i = \mathbf{l}\mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{l}\mathbf{F} \\ \dot{\mathbf{F}}^{-1} &= \mathbf{G}_i \otimes \dot{\mathbf{g}}^i = -\mathbf{G}_i \otimes \mathbf{g}^i \mathbf{l} = -\mathbf{F}^{-1}\mathbf{l} \\ \dot{\mathbf{F}}^{-\mathrm{T}} &= \dot{\mathbf{g}}^i \otimes \mathbf{G}_i = -\mathbf{g}^i \mathbf{l} \otimes \mathbf{G}_i = -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \otimes \mathbf{G}_i = -\mathbf{l}^{\mathrm{T}} \mathbf{F}^{-\mathrm{T}} \\ \dot{\mathbf{F}}^{\mathrm{T}} &= \mathbf{G}^i \otimes \dot{\mathbf{g}}_i = \mathbf{G}^i \otimes \mathbf{g}_i \mathbf{l}^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}} \mathbf{l}^{\mathrm{T}} \end{aligned} \qquad (2.11.9)$$

which are Eqns. 2.5.4-5.

An alternative way of arriving at Eqns. 2.11.7 is to start with Eqns. 2.11.9: the covariant base vectors $\mathbf{G}_i$ convect to $\mathbf{g}_i(t)$ over time through the time-dependent deformation gradient: $\mathbf{g}_i(t) = \mathbf{F}(t)\mathbf{G}_i$. For this relation to hold at all times, one must have, from Eqn. 2.11.9b,

$$\begin{aligned} \dot{\mathbf{G}}_i = 0 &= \overline{\mathbf{F}^{-1}\mathbf{g}_i}^{\boldsymbol{\cdot}} \\ &= \dot{\mathbf{F}}^{-1}\mathbf{g}_i + \mathbf{F}^{-1}\dot{\mathbf{g}}_i \\ &= \mathbf{F}^{-1}\left(-\mathbf{l}\mathbf{g}_i + \dot{\mathbf{g}}_i\right) \end{aligned} \qquad (2.11.10)$$

Thus, in order to maintain the convection of the tangent basis over time, one requires that

$$\dot{\mathbf{g}}_i = \mathbf{l}\mathbf{g}_i \tag{2.11.11}$$

The contravariant base vectors $\mathbf{G}^i$ transform to $\mathbf{g}^i(t)$ over time through the time-dependent inverse transpose of the deformation gradient: $\mathbf{g}^i(t) = \mathbf{F}^{-\mathrm{T}}(t)\mathbf{G}^i$. For this relation to hold at all times, one must have, from Eqn. 2.11.9d,

$$\begin{aligned}\dot{\mathbf{G}}^i = 0 &= \overline{\mathbf{F}^{\mathrm{T}}\mathbf{g}^i}^{\,\bullet} \\ &= \dot{\mathbf{F}}^{\mathrm{T}}\mathbf{g}^i + \mathbf{F}^{\mathrm{T}}\dot{\mathbf{g}}^i \\ &= \mathbf{F}^{\mathrm{T}}\left(\mathbf{l}^{\mathrm{T}}\mathbf{g}^i + \dot{\mathbf{g}}^i\right)\end{aligned} \tag{2.11.12}$$

Thus, in order to maintain the convection of the normal basis over time, one requires that

$$\dot{\mathbf{g}}^i = -\mathbf{l}^{\mathrm{T}}\mathbf{g}^i \tag{2.11.13}$$

The Rate of Deformation and Spin Tensors

From 2.5.6, $\mathbf{l} = \mathbf{d} + \mathbf{w}$. The covariant components of the rate of deformation and spin are

$$\begin{aligned}d_{ij} &= \frac{1}{2}\mathbf{g}_i\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right)\mathbf{g}_j = \frac{1}{2}\mathbf{g}_i\left(\dot{\mathbf{g}}_m \otimes \mathbf{g}^m + \mathbf{g}^m \otimes \dot{\mathbf{g}}_m\right)\mathbf{g}_j = \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right) = \frac{1}{2}\overline{\mathbf{g}_i \cdot \mathbf{g}_j}^{\,\bullet} \\ w_{ij} &= \frac{1}{2}\mathbf{g}_i\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right)\mathbf{g}_j = \frac{1}{2}\mathbf{g}_i\left(\dot{\mathbf{g}}_m \otimes \mathbf{g}^m - \mathbf{g}^m \otimes \dot{\mathbf{g}}_m\right)\mathbf{g}_j = \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j - \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right)\end{aligned} \tag{2.11.14}$$

Alternatively, from 2.11.6a,

$$\begin{aligned}\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right) &= \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right)\mathbf{g}_i \otimes \mathbf{g}_j \\ &= \frac{1}{2}\overline{\mathbf{g}_i \cdot \mathbf{g}_j}^{\,\bullet}\,\mathbf{g}_i \otimes \mathbf{g}_j \\ &= \frac{1}{2}\dot{g}_{ij}\mathbf{g}_i \otimes \mathbf{g}_j\end{aligned} \tag{2.11.15}$$

2.12 Pull Back, Push Forward and Lie Time Derivatives

This section is in the main concerned with the following issue: an observer attached to a fixed, say Cartesian, coordinate system will see a material move and deform over time, and will observe various vectorial and tensorial quantities to change also. However, a hypothetical observer attached to the deforming material, and moving and deforming with the material, will see something different. The question is: what quantities will be seen to change from this embedded observer's viewpoint?

2.12.1 Time Derivatives of Spatial Fields

In terms of the spatial basis, a spatial vector **v** can be expressed in terms of the covariant components and contravariant components,

$$\mathbf{v} = v_i \mathbf{g}^i, \qquad \mathbf{v} = v^i \mathbf{g}_i \tag{2.12.1}$$

We want to distinguish between two quantities. The first is the material time derivative of the vector **v**:

$$\dot{\mathbf{v}} = \dot{\overline{v_i \mathbf{g}^i}} = \dot{v}_i \mathbf{g}^i + v_i \dot{\mathbf{g}}^i, \qquad \dot{\mathbf{v}} = \dot{\overline{v^i \mathbf{g}_i}} = \dot{v}^i \mathbf{g}_i + v^i \dot{\mathbf{g}}_i \tag{2.12.2}$$

The second is the time derivative *holding the base vectors fixed*,

$$\dot{v}_i \mathbf{g}^i, \qquad \dot{v}^i \mathbf{g}_i \tag{2.12.3}$$

This latter is called the **convected derivative** and is the rate of the change as seen by an observer attached to the deforming bases.

From Eqn. 2.12.1, the components of **v** can be expressed as

$$v_i = \mathbf{v} \cdot \mathbf{g}_i, \quad v^i = \mathbf{v} \cdot \mathbf{g}^i \tag{2.12.4}$$

Taking the material time derivative, and using Eqns. 2.11.11, 2.11.13,

$$\begin{aligned} \dot{v}_i &= \dot{\overline{\mathbf{v} \cdot \mathbf{g}_i}} & \qquad \dot{v}^i &= \dot{\overline{\mathbf{v} \cdot \mathbf{g}^i}} \\ &= \dot{\mathbf{v}} \cdot \mathbf{g}_i + \mathbf{v} \cdot \dot{\mathbf{g}}_i\,, & &= \dot{\mathbf{v}} \cdot \mathbf{g}^i + \mathbf{v} \cdot \dot{\mathbf{g}}^i \\ &= \left(\dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v}\right) \cdot \mathbf{g}_i & &= \left(\dot{\mathbf{v}} - \mathbf{l}\mathbf{v}\right) \cdot \mathbf{g}^i \end{aligned} \tag{2.12.5}$$

Thus there are two convected derivatives of a vector, depending on whether one is using covariant or contravariant components:

$$\dot{v}_i \mathbf{g}^i = \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v}$$
$$\dot{v}^i \mathbf{g}_i = \dot{\mathbf{v}} - \mathbf{l}\mathbf{v} \tag{2.12.6}$$

As will be seen below, these quantities are **Lie derivatives** of the vector **v**.

The time derivative of the components can be expressed in an alternative way, by expressing the spatial base vectors $\mathbf{g}_i, \mathbf{g}^i$ in terms of the material base vectors $\mathbf{G}_i, \mathbf{G}^i$; using Eqns. 2.10.23:

$$\begin{aligned} \dot{v}_i &= \overline{\mathbf{v} \cdot \mathbf{g}_i}^{\,\cdot} & \dot{v}^i &= \overline{\mathbf{v} \cdot \mathbf{g}^i}^{\,\cdot} \\ &= \overline{\mathbf{v} \cdot \mathbf{F}\mathbf{G}_i}^{\,\cdot}, & &= \overline{\mathbf{v} \cdot \mathbf{F}^{-\mathrm{T}}\mathbf{G}^i}^{\,\cdot} \\ &= \overline{\mathbf{F}^{\mathrm{T}}\mathbf{v}}^{\,\cdot}\mathbf{G}_i & &= \overline{\mathbf{F}^{-1}\mathbf{v}}^{\,\cdot}\mathbf{G}^i \end{aligned} \tag{2.12.7}$$

So, as an alternative to Eqns. 2.12.6,

$$\dot{v}_i \mathbf{G}^i = \overline{\mathbf{F}^{\mathrm{T}}\mathbf{v}}^{\,\cdot}$$
$$\dot{v}^i \mathbf{G}_i = \overline{\mathbf{F}^{-1}\mathbf{v}}^{\,\cdot} \tag{2.12.8}$$

As will be seen further below, the quantities on the right are the material time derivatives of the **pull-back** of the vector **v**.

Repeating the above, now for a spatial tensor **a**: in terms of the spatial basis, **a** can be expressed in terms of the covariant components and contravariant components as

$$\mathbf{a} = a_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \qquad \mathbf{a} = a^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \tag{2.12.9}$$

The material time derivative of the tensor **a** is

$$\begin{aligned} \dot{\mathbf{a}} &= \overline{a_{ij} \mathbf{g}^i \otimes \mathbf{g}^j}^{\,\cdot} = \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j + a_{ij} \dot{\mathbf{g}}^i \otimes \mathbf{g}^j + a_{ij} \mathbf{g}^i \otimes \dot{\mathbf{g}}^j \\ &= \overline{a^{ij} \mathbf{g}_i \otimes \mathbf{g}_j}^{\,\cdot} = \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j + a^{ij} \dot{\mathbf{g}}_i \otimes \mathbf{g}_j + a^{ij} \mathbf{g}_i \otimes \dot{\mathbf{g}}_j \end{aligned} \tag{2.12.10}$$

and the convected derivative is the first term:

$$\dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \qquad \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \tag{2.12.11}$$

The components of **a** can be expressed as

$$a_{ij} = \mathbf{g}_i \mathbf{A} \mathbf{g}_j, \quad a^{ij} = \mathbf{g}^i \mathbf{A} \mathbf{g}^j \tag{2.12.12}$$

Taking the material time derivative, and again using Eqns. 2.11.11, 2.11.13,

$$\begin{aligned} \dot{a}_{ij} &= \dot{\overline{\mathbf{g}_i \mathbf{a} \mathbf{g}_j}} \\ &= \dot{\mathbf{g}}_i \mathbf{a} \mathbf{g}_j + \mathbf{g}_i \dot{\mathbf{a}} \mathbf{g}_j + \mathbf{g}_i \mathbf{a} \dot{\mathbf{g}}_j \,, \\ &= \mathbf{g}_i \left(\dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}} \mathbf{a} \right) \mathbf{g}_j \end{aligned} \qquad \begin{aligned} \dot{a}^{ij} &= \dot{\overline{\mathbf{g}^i \mathbf{a} \mathbf{g}^j}} \\ &= -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \mathbf{a} \mathbf{g}^j + \mathbf{g}^i \dot{\mathbf{a}} \mathbf{g}^j - \mathbf{g}^i \mathbf{a} \mathbf{l}^{\mathrm{T}} \mathbf{g}^j \\ &= \mathbf{g}^i \left(\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} \right) \mathbf{g}^j \end{aligned} \tag{2.12.13}$$

The convected derivatives are thus

$$\begin{aligned} \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j &= \dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}} \mathbf{a} \\ \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j &= \dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} \end{aligned} \tag{2.12.14}$$

As will be seen below, these quantities are **Lie derivatives** of the tensor **a**.

The time derivative of the components can be expressed in an alternative way, by expressing the spatial base vectors $\mathbf{g}_i, \mathbf{g}^i$ in terms of the material base vectors $\mathbf{G}_i, \mathbf{G}^i$; using Eqns. 2.10.23:

$$\begin{aligned} \dot{a}_{ij} &= \dot{\overline{\mathbf{g}_i \mathbf{a} \mathbf{g}_j}} \\ &= \dot{\overline{\mathbf{F}\mathbf{G}_i \mathbf{a} \mathbf{F}\mathbf{G}_i}} \quad , \\ &= \mathbf{G}_i \, \dot{\overline{\mathbf{F}^{\mathrm{T}} \mathbf{a} \mathbf{F}}} \, \mathbf{G}_i \end{aligned} \qquad \begin{aligned} \dot{a}^{ij} &= \dot{\overline{\mathbf{g}^i \mathbf{a} \mathbf{g}^j}} \\ &= \dot{\overline{\mathbf{F}^{-\mathrm{T}} \mathbf{G}^i \mathbf{a} \mathbf{F}^{-\mathrm{T}} \mathbf{G}^j}} \\ &= \mathbf{G}^i \, \dot{\overline{\mathbf{F}^{-1} \mathbf{a} \mathbf{F}^{-\mathrm{T}}}} \, \mathbf{G}^j \end{aligned} \tag{2.12.15}$$

So, as an alternative to Eqns. 2.12.14,

$$\begin{aligned} \dot{a}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j &= \dot{\overline{\mathbf{F}^{\mathrm{T}} \mathbf{a} \mathbf{F}}} \\ \dot{a}^{ij} \mathbf{G}_i \otimes \mathbf{G}_j &= \dot{\overline{\mathbf{F}^{-1} \mathbf{a} \mathbf{F}^{-\mathrm{T}}}} \end{aligned} \tag{2.12.16}$$

As will be seen next, the quantities on the right are the material time derivatives of the **pull-back** of the tensor **a**.

Example

Considering again Example 1 which was worked through in detail in §2.10, suppose we have a shearing deformation as shown in Fig. 2.12.1 (this is Fig. 2.10.3).

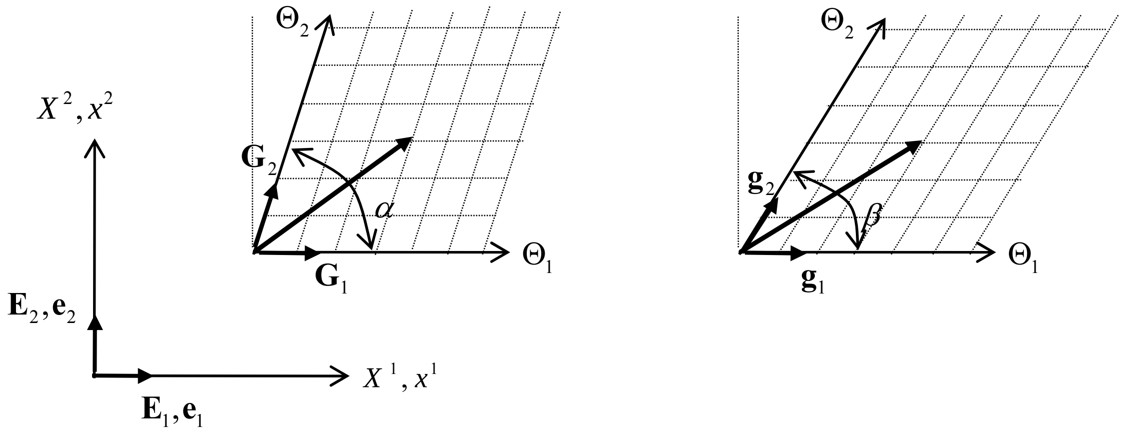


Figure 2.12.1: A pure shear deformation of one parallelogram into another

Let the shear angle β in Fig. 2.12.1 evolve over time according to

$$\beta = \alpha + \gamma t \tag{2.12.17}$$

From Eqns. 2.10.7, 2.10.11, the rates of change of the base vectors are

$$\begin{aligned} &\frac{d}{dt}\mathbf{g}_1 = \frac{d}{dt}\mathbf{e}_1 = 0, \qquad \frac{d}{dt}\mathbf{g}_2 = \frac{d}{dt}\left(\frac{1}{\tan(\alpha+\gamma t)}\mathbf{e}_1 + \mathbf{e}_2\right) = -\frac{\gamma}{\sin^2(\alpha+\gamma t)}\mathbf{e}_1 \\ &\frac{d}{dt}\mathbf{g}^1 = \frac{d}{dt}\left(\mathbf{e}_1 - \frac{1}{\tan(\alpha+\gamma t)}\mathbf{e}_2\right) = +\frac{\gamma}{\sin^2(\alpha+\gamma t)}\mathbf{e}_2, \qquad \frac{d}{dt}\mathbf{g}^2 = \frac{d}{dt}\mathbf{e}_2 = 0 \end{aligned} \tag{2.12.18}$$

The velocity gradient is, from Eqn. 2.11.5,

$$\begin{aligned} \mathbf{l} &= \dot{\mathbf{g}}_1 \otimes \mathbf{g}^1 + \dot{\mathbf{g}}_2 \otimes \mathbf{g}^2 \\ &= -\frac{\gamma}{\sin^2\beta}\mathbf{e}_1 \otimes \mathbf{e}_2 \\ &= \dot{\Pi}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (\mathbf{e}_i) \end{aligned} \tag{2.12.19}$$

where Π is given by Eqn. 2.10.26, and

$$\dot{\Pi}(t) = \frac{d}{dt}\left(\frac{1}{\tan(\alpha+\gamma t)} - \frac{1}{\tan(\alpha)}\right) = -\frac{\gamma}{\sin^2(\alpha+\gamma t)} \tag{2.12.20}$$

Considering again the vector **V** of Eqn. 2.10.44, $\mathbf{V} = \begin{bmatrix} V_x & V_y \end{bmatrix}^{\mathrm{T}} \ (\mathbf{E}_i)$, and its corresponding deformed vector **v** of Eqn. 2.10.47, $\mathbf{v} = \begin{bmatrix} V_x + \Pi V_y & V_y \end{bmatrix} (\mathbf{e}_i)$,

$$\dot{\mathbf{v}} = \dot{\Pi} \begin{bmatrix} V_y \\ 0 \end{bmatrix} \ (\mathbf{e}_i), \tag{2.12.21}$$

The contravariant and covariant components of $\dot{\mathbf{v}}$ are

$$\dot{\mathbf{v}} = \widehat{v}^i \mathbf{g}_i, \quad \widehat{v}^i = \dot{\Pi} \begin{bmatrix} V_y \\ 0 \end{bmatrix}, \qquad \dot{\mathbf{v}} = \widehat{v}_i \mathbf{g}^i, \quad \widehat{v}_i = \dot{\Pi} \begin{bmatrix} V_y \\ \dfrac{1}{\tan\beta} V_y \end{bmatrix} \tag{2.12.22}$$

The "hat" on the $\widehat{v}$ is to emphasise that (see Eqns. 2.12.5)

$$\widehat{v}^i = \dot{\mathbf{v}} \cdot \mathbf{g}^i \neq \dot{v}^i = \dot{\overline{\mathbf{v} \cdot \mathbf{g}^i}}, \qquad \widehat{v}_i = \dot{\mathbf{v}} \cdot \mathbf{g}_i \neq \dot{v}_i = \dot{\overline{\mathbf{v} \cdot \mathbf{g}_i}} \tag{2.12.23}$$

From Eqns. 2.12.6, the convected derivatives are

$$\begin{aligned} \dot{\mathbf{v}} - \mathbf{l}\mathbf{v} &= \dot{\Pi} \left\{ \begin{bmatrix} V_y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_x + \Pi V_y \\ V_y \end{bmatrix} \right\}, & \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}}\mathbf{v} &= \dot{\Pi} \left\{ \begin{bmatrix} V_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_x + \Pi V_y \\ V_y \end{bmatrix} \right\} \\ &= \dot{\Pi} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & &= \dot{\Pi} \begin{bmatrix} V_y \\ V_x + \Pi V_y \end{bmatrix} \end{aligned} \tag{2.12.23}$$

Thus $\dot{\mathbf{v}} - \mathbf{l}\mathbf{v} = 0$, which, from Eqn. 2.12.6, implies that $\dot{v}^i = 0$. This is the expected result: the contravariant components do not change over time. They are always $\begin{bmatrix} V_x - V_y / \tan\alpha & V_y \end{bmatrix}$, as given by Eqn. 2.10.47b.

Consider now an example tensor

$$\mathbf{A} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \ (\mathbf{E}_i) \tag{2.12.24}$$

The covariant and contravariant components are

$$\mathbf{A} = \begin{bmatrix} A_{xx} - \frac{1}{\tan\alpha}\left(A_{xy} + A_{yx}\right) + \frac{1}{\tan^2\alpha}A_{yy} & A_{xy} - \frac{1}{\tan\alpha}A_{yy} \\ A_{yx} - \frac{1}{\tan\alpha}A_{yy} & A_{yy} \end{bmatrix} \quad (\mathbf{G}_i)$$

$$\mathbf{A} = \begin{bmatrix} A_{xx} & A_{xy} + A_{xx}\frac{1}{\tan\alpha} \\ A_{yx} + A_{xx}\frac{1}{\tan\alpha} & A_{yy} + \frac{1}{\tan\alpha}\left(A_{xy} + A_{yx}\right) + \frac{1}{\tan^2\alpha}A_{xx} \end{bmatrix} \quad (\mathbf{G}^i) \tag{2.12.25}$$

This deforms to (with **F** given by Eqn. 2.10.25)

$$\mathbf{a} = \begin{bmatrix} 1 & \Pi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} = \begin{bmatrix} A_{xx} + \Pi A_{yx} & A_{xy} + \Pi A_{yy} \\ A_{yx} & A_{yy} \end{bmatrix} \quad (\mathbf{e}_i) \tag{2.12.25}$$

Now

$$\begin{aligned} \mathbf{FA} &= \left(\mathbf{g}_i \otimes \mathbf{G}^i\right) A^{mn} \mathbf{G}_m \otimes \mathbf{G}_n \\ &= A^{ij} \mathbf{g}_i \otimes \mathbf{G}_j \end{aligned} \tag{2.12.26}$$

Converting between the various convected base vectors using Eqns. 2.10.7-8, 2.10.11-12, the contravariant and covariant and components are $\mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$, $\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$:

$$a^{ij} = \begin{bmatrix} A_{xx} - \frac{1}{\tan\beta}A_{xy} - \frac{1}{\tan\alpha}A_{yx} + \frac{1}{\tan\alpha}\frac{1}{\tan\beta}A_{yy} & A_{xy} - A_{yy}\frac{1}{\tan\alpha} \\ A_{yx} - \frac{1}{\tan\beta}A_{yy} & A_{yy} \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} A_{xx} + A_{yx}\Pi & A_{xy} + A_{yy}\Pi + \frac{1}{\tan\beta}\left(A_{xx} + A_{yx}\Pi\right) \\ A_{yx} + \frac{1}{\tan\beta}\left(A_{xx} + A_{yx}\Pi\right) & A_{yy} + \frac{1}{\tan\beta}A_{yx} + \frac{1}{\tan\beta}\left(A_{xy} + A_{yy}\Pi\right) + \frac{1}{\tan^2\beta}\left(A_{xx} + A_{yx}\Pi\right) \end{bmatrix} \tag{2.12.27}$$

Also,

$$\dot{\mathbf{a}} = \dot{\Pi}\begin{bmatrix} A_{yx} & A_{yy} \\ 0 & 0 \end{bmatrix} \quad (\mathbf{e}_i), \tag{2.12.28}$$

and the contravariant and covariant components are

$$\dot{\mathbf{a}} = \hat{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j, \quad \hat{a}^{ij} = \dot{\Pi}\begin{bmatrix} A_{yx} - \dfrac{1}{\tan\beta}A_{yy} & A_{yy} \\ 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{a}} = \hat{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \quad \hat{a}_{ij} = \dot{\Pi}\begin{bmatrix} A_{yx} & \dfrac{1}{\tan\beta}A_{yx} + A_{yy} \\ \dfrac{1}{\tan\beta}A_{yx} & \dfrac{1}{\tan\beta}\left(\dfrac{1}{\tan\beta}A_{yx} + A_{yy}\right) \end{bmatrix} \tag{2.12.29}$$

Again, the "hat" emphasises that (see Eqns. 2.12.13)

$$\hat{a}^{ij} = \mathbf{g}^i\dot{\mathbf{a}}\mathbf{g}^j \neq \dot{a}^{ij} = \overline{\mathbf{g}^i\mathbf{a}\mathbf{g}^j}^{\,\cdot}, \qquad \hat{a}_{ij} = \mathbf{g}_i\dot{\mathbf{a}}\mathbf{g}_j \neq \dot{a}_{ij} = \overline{\mathbf{g}_i\mathbf{a}\mathbf{g}_j}^{\,\cdot} \tag{2.12.30}$$

Now

$$\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} = \dot{\Pi}\begin{bmatrix} -A_{xy} - A_{yy}\Pi & 0 \\ -A_{yy} & 0 \end{bmatrix}$$

$$\dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}}\mathbf{a} = \dot{\Pi}\begin{bmatrix} A_{yx} & A_{xx} + A_{yx}\Pi + A_{yy} \\ A_{xx} + A_{yx}\Pi & A_{xy} + A_{yx} + A_{yy}\Pi \end{bmatrix} \tag{2.12.31}$$

Thus $\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} = 0$, i.e. $\dot{a}^{ij} = 0$, only when $A_{xy} = A_{yy} = 0$, which is consistent with Eqn. 2.12.27a (only constant terms, independent of β remain in that case).

2.12.2 Push-Forward and Pull-Back

Next are defined the push-forward and pull-back of vectors and tensors, which will lead into the concept of Lie derivatives, which relate back to what was just discussed above regarding convected derivatives.

Vectors

Consider a vector **V** given in terms of the reference configuration base vectors:

$$\begin{aligned} \mathbf{V} &= V_i\left(\Theta^j\right)\mathbf{G}^i \\ &= V^i\left(\Theta^j\right)\mathbf{G}_i \end{aligned} \tag{2.12.32}$$

The **push-forward**, symbolised by $\chi_*(\bullet)$, is defined to be the vector with *the same components*, but with respect to the current configuration base vectors. There are 2 push-forward operations, depending on the type of components used; the symbol b is used for covariant components V_i and the symbol # for contravariant components V^i; using 2.10.23,

$$\boxed{\begin{aligned}\chi_*(\mathbf{V})^b &\equiv V_i\mathbf{g}^i = V_i\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i = \mathbf{F}^{-\mathrm{T}}\mathbf{V}\\ \chi_*(\mathbf{V})^\# &\equiv V^i\mathbf{g}_i = V^i\mathbf{F}\mathbf{G}_i = \mathbf{F}\mathbf{V}\end{aligned}} \quad \textbf{Push-forward of Vector} \qquad (2.12.33)$$

Eqn. 2.12.33b says that the push forward of the contravariant form of **V** is simply **FV**. In other words, the push forward here is the actual corresponding vector in the deformed configuration, $\mathbf{v} = \mathbf{F}\mathbf{V} = v^i(\Theta^j)\mathbf{g}_i$, and, as a consequence of the definitions, $V^i = v^i$, as illustrated in Fig. 2.12.2.

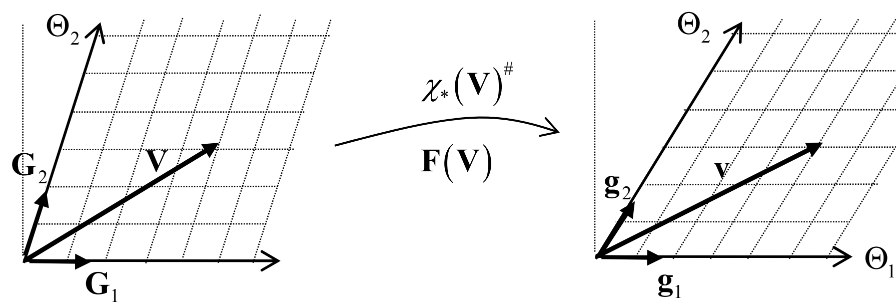


Figure 2.12.2: The push-forward of a vector V

A special case of Eqn. 2.12.33b is the push forward of a line element in the reference configuration, giving the corresponding line element in the current configuration:

$$\chi_*(d\mathbf{X})^\# = d\Theta^i\mathbf{g}_i = d\mathbf{x}\,. \qquad (2.12.34)$$

Similarly, consider a vector **v** given in terms of the current configuration basis:

$$\mathbf{v} = v_i\mathbf{g}^i = v^i\mathbf{g}_i \qquad (2.12.35)$$

The **pull-back** of **v**, $\chi_*^{-1}(\mathbf{v})$, is defined to be the vector with components v_i (or v^i) with respect to the reference configuration base vectors $\mathbf{G}^i$ (or $\mathbf{G}_i$). Using 2.10.23,

$$\boxed{\begin{aligned}\chi_*^{-1}(\mathbf{v})^b &= v_i\mathbf{G}^i = v_i\mathbf{F}^{\mathrm{T}}\mathbf{g}^i = \mathbf{F}^{\mathrm{T}}\mathbf{v}\\ \chi_*^{-1}(\mathbf{v})^\# &= v^i\mathbf{G}_i = v^i\mathbf{F}^{-1}\mathbf{g}_i = \mathbf{F}^{-1}\mathbf{v}\end{aligned}} \quad \textbf{Pull-back of a vector} \qquad (2.12.36)$$

and, for a line element in the current configuration,

$$\chi_*^{-1}(d\mathbf{x})^\# = dx^i\mathbf{G}_i = \mathbf{F}^{-1}d\mathbf{x} = d\mathbf{X}\,. \qquad (2.12.37)$$

Note that a push-forward and pull-back applied successively to a vector with the same component type will result in the initial vector.

From the above, for two material vectors **U** and **V** and two spatial vectors **u** and **v**,

$$\begin{aligned}\mathbf{U}\cdot\mathbf{V} &= \chi_*(\mathbf{U})^b \cdot \chi_*(\mathbf{V})^\# = \chi_*(\mathbf{U})^\# \cdot \chi_*(\mathbf{V})^b \\ \mathbf{u}\cdot\mathbf{v} &= \chi_*^{-1}(\mathbf{u})^b \cdot \chi_*^{-1}(\mathbf{v})^\# = \chi_*^{-1}(\mathbf{u})^\# \cdot \chi_*^{-1}(\mathbf{v})^b\end{aligned} \tag{2.12.38}$$

For example, as a special case of this, in the reference configuration, $\mathbf{G}_1$ and $\mathbf{G}^2$ are perpendicular: $\mathbf{G}_1 \cdot \mathbf{G}^2 = 0$. Pushing forward these vectors, we get from Eqn. 2.12.33: $\mathbf{F}\mathbf{G}_1 = \mathbf{g}_1$ and $\mathbf{F}^{-\mathrm{T}}\mathbf{G}^2 = \mathbf{g}^2$, and again $\chi_*(\mathbf{G}_1)^\# \cdot \chi_*(\mathbf{G}^2)^b = \mathbf{g}_1 \cdot \mathbf{g}^2 = 0$.

Tensors

Consider a material tensor **A**:

$$\mathbf{A} = A_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = A^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = A^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = A_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j \tag{2.12.39}$$

As for the vector, the push-forward of **A**, $\chi_*(\mathbf{A})$, is defined to be the tensor with the same components, but with respect to the deformed base vectors. Thus, using 2.10.23,

$$\boxed{\begin{aligned}\chi_*(\mathbf{A})^b &= A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = A_{ij}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{F}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{-1} \\ \chi_*(\mathbf{A})^\# &= A^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = A^{ij}\left(\mathbf{F}\mathbf{G}_i \otimes \mathbf{F}\mathbf{G}_j\right) = \mathbf{F}\mathbf{A}\mathbf{F}^{\mathrm{T}} \\ \chi_*(\mathbf{A})^\backslash &= A^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = A^i_{\cdot j}\left(\mathbf{F}\mathbf{G}_i \otimes \mathbf{F}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{F}\mathbf{A}\mathbf{F}^{-1} \\ \chi_*(\mathbf{A})^/ &= A_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = A_i^{\cdot j}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{F}\mathbf{G}_j\right) = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{\mathrm{T}}\end{aligned}} \quad \textbf{Push-forward of Tensor} \tag{2.12.40}$$

Similarly, consider a spatial tensor **a**:

$$\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = a^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = a_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j \tag{2.12.41}$$

The pull-back is

$$\begin{aligned}
\chi_*^{-1}(\mathbf{a})^{b} &= a_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = a_{ij}\left(\mathbf{F}^{\mathrm{T}}\mathbf{g}^i \otimes \mathbf{F}^{\mathrm{T}}\mathbf{g}^j\right) &= \mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F} \\
\chi_*^{-1}(\mathbf{a})^{\#} &= a^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = a^{ij}\left(\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{-1}\mathbf{g}_j\right) &= \mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}} \\
\chi_*^{-1}(\mathbf{a})^{\backslash} &= a^{i}_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = a^{i}_{\cdot j}\left(\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{\mathrm{T}}\mathbf{g}^j\right) &= \mathbf{F}^{-1}\mathbf{a}\mathbf{F} \\
\chi_*^{-1}(\mathbf{a})^{/} &= a_{i}^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = a_{i}^{\cdot j}\left(\mathbf{F}^{\mathrm{T}}\mathbf{g}^i \otimes \mathbf{F}^{-1}\mathbf{g}_j\right) &= \mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}^{-\mathrm{T}}
\end{aligned}$$

Pull-back of Tensor (2.12.42)

The first of these, $\mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}$, is called the **covariant pull-back**, whereas the second, $\mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}}$, is called the **contravariant pull-back**.

Since $\mathbf{F}$ maps material vectors to spatial vectors, $\mathbf{a}$ maps spatial vectors to spatial vectors, and $\mathbf{F}^{\mathrm{T}}$ maps spatial vectors to material vectors, it follows that the pull-back $\mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}$ maps material vectors to material vectors, and so is a material tensor field, and similarly for the other three pull-backs.

Time Derivatives

It will be recognised that the expressions for the pull backs of a spatial covariant tensor and spatial contravariant tensor in Eqns. 2.12.42a,b are those appearing in Eqns. 2.12.16. Keeping in mind Eqn. 2.12.14, one sees that, for a spatial tensor in terms of covariant components, $\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, and contravariant components, $\mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$,

$$\begin{aligned}
\dot{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j &= \left(\overline{\mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}}^{\cdot}\right)\mathbf{g}^i \otimes \mathbf{g}^j &= \dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}}\mathbf{a} \\
\dot{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j &= \left(\overline{\mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}}}^{\cdot}\right)\mathbf{g}_i \otimes \mathbf{g}_j &= \dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}}
\end{aligned} \qquad (2.12.43)$$

Other Push-Forward and Pull-Back relations for Vectors and Tensors

Here follow some relations involving the push-forward and pull-backs of tensors.

For two material tensors $\mathbf{A}$ and $\mathbf{B}$ and two spatial tensors $\mathbf{a}$ and $\mathbf{b}$, the scalar product is

$$\begin{aligned}
\mathbf{A}:\mathbf{B} &= A_{ij}B^{ij} = A^{ij}B_{ij} = A^{i}_{\cdot j}B_{i}^{\cdot j} = A_{i}^{\cdot j}B^{i}_{\cdot j} \\
\mathbf{a}:\mathbf{b} &= a_{ij}b^{ij} = a^{ij}b_{ij} = a^{i}_{\cdot j}b_{i}^{\cdot j} = a_{i}^{\cdot j}b^{i}_{\cdot j}
\end{aligned} \qquad (2.12.44)$$

This scalar product then push-forwards and pull-backs as {▲Problem 1}

$$\begin{aligned}\mathbf{A}:\mathbf{B} &= \chi_*(\mathbf{A})^b : \chi_*(\mathbf{B})^\# = \chi_*(\mathbf{A})^\# : \chi_*(\mathbf{B})^b \\ &= \chi_*(\mathbf{A})^/ : \chi_*(\mathbf{B})^\backslash = \chi_*(\mathbf{A})^\backslash : \chi_*(\mathbf{B})^/ \\ \mathbf{a}:\mathbf{b} &= \chi_*^{-1}(\mathbf{a})^b : \chi_*^{-1}(\mathbf{b})^\# = \chi_*^{-1}(\mathbf{a})^\# : \chi_*^{-1}(\mathbf{b})^b \\ &= \chi_*^{-1}(\mathbf{a})^/ : \chi_*^{-1}(\mathbf{b})^\backslash = \chi_*^{-1}(\mathbf{a})^\backslash : \chi_*^{-1}(\mathbf{b})^/\end{aligned} \tag{2.12.45}$$

For material tensor **A** and material vectors $\mathbf{U}, \mathbf{V}$, and spatial tensor **a** and spatial vectors $\mathbf{u}, \mathbf{v}$,

$$\begin{aligned}\mathbf{UAV} &= U_i A^{ij} V_j = U^i A_{ij} V^j = U_i A^i_{\cdot j} V^j = U^i A_i^{\cdot j} V_j \\ \mathbf{uav} &= u_i a^{ij} v_j = u^i a_{ij} v^j = u_i a^i_{\cdot j} v^j = u^i a_i^{\cdot j} v_j\end{aligned} \tag{2.12.46}$$

Then

$$\begin{aligned}\mathbf{UAV} &= \chi_*(\mathbf{U})^b \chi_*(\mathbf{A})^\# \chi_*(\mathbf{V})^b = \chi_*(\mathbf{U})^\# \chi_*(\mathbf{A})^b \chi_*(\mathbf{V})^\# \\ &= \chi_*(\mathbf{U})^b \chi_*(\mathbf{A})^\backslash \chi_*(\mathbf{V})^\# = \chi_*(\mathbf{U})^\# \chi_*(\mathbf{A})^/ \chi_*(\mathbf{V})^b \\ \mathbf{uav} &= \chi_*^{-1}(\mathbf{u})^b \chi_*^{-1}(\mathbf{a})^\# \chi_*^{-1}(\mathbf{v})^b = \chi_*^{-1}(\mathbf{u})^\# \chi_*^{-1}(\mathbf{a})^b \chi_*^{-1}(\mathbf{v})^\# \\ &= \chi_*^{-1}(\mathbf{u})^b \chi_*^{-1}(\mathbf{a})^\backslash \chi_*^{-1}(\mathbf{v})^\# = \chi_*^{-1}(\mathbf{u})^\# \chi_*^{-1}(\mathbf{a})^/ \chi_*^{-1}(\mathbf{v})^b\end{aligned} \tag{2.12.47}$$

For material tensor **A** and material vector **V**, and spatial tensor **a** and spatial vector **v**, the contractions $\mathbf{AV}$ and $\mathbf{av}$ are

$$\begin{aligned}\mathbf{AV} &= A_{ij} V^j = A_i^{\cdot j} V_j = A^i_{\cdot j} V^j = A^{ij} V_j \\ \mathbf{av} &= a_{ij} v^j = a_i^{\cdot j} v_j = a^i_{\cdot j} v^j = a^{ij} v_j\end{aligned} \tag{2.12.48}$$

and so transform as

$$\begin{aligned}\chi_*(\mathbf{AV})^b &= \chi_*(\mathbf{A})^b \chi_*(\mathbf{V})^\# = \chi_*(\mathbf{A})^/ \chi_*(\mathbf{V})^b \\ \chi_*(\mathbf{AV})^\# &= \chi_*(\mathbf{A})^\# \chi_*(\mathbf{V})^b = \chi_*(\mathbf{A})^\backslash \chi_*(\mathbf{V})^\# \\ \\ \chi_*^{-1}(\mathbf{av})^b &= \chi_*^{-1}(\mathbf{a})^b \chi_*^{-1}(\mathbf{v})^\# = \chi_*^{-1}(\mathbf{a})^/ \chi_*^{-1}(\mathbf{v})^b \\ \chi_*^{-1}(\mathbf{av})^\# &= \chi_*^{-1}(\mathbf{a})^\# \chi_*^{-1}(\mathbf{v})^b = \chi_*^{-1}(\mathbf{a})^\backslash \chi_*^{-1}(\mathbf{v})^\#\end{aligned} \tag{2.12.49}$$

Finally, for material tensors **A**, **B** and spatial tensors **a**, **b**,

$$\begin{aligned}\mathbf{AB} &= A_{ik} B^{kj} \mathbf{G}^i \otimes \mathbf{G}_j = A_i^{\cdot k} B_k^{\cdot j} \mathbf{G}^i \otimes \mathbf{G}_j = A_i^{\cdot k} B_{kj} \mathbf{G}^i \otimes \mathbf{G}^j = A_{ik} B^k_{\cdot j} \mathbf{G}^i \otimes \mathbf{G}^j = \cdots \\ \mathbf{ab} &= a_{ik} b^{kj} \mathbf{g}^i \otimes \mathbf{g}_j = a_i^{\cdot k} b_k^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j = a_i^{\cdot k} b_{kj} \mathbf{g}^i \otimes \mathbf{g}^j = a_{ik} b^k_{\cdot j} \mathbf{g}^i \otimes \mathbf{g}^j = \cdots\end{aligned} \tag{2.12.50}$$

and so

$$
\begin{aligned}
&\chi_*(\mathbf{AB})^{/} = \chi_*(\mathbf{A})^{b}\,\chi_*(\mathbf{B})^{\#} = \chi_*(\mathbf{A})^{/}\,\chi_*(\mathbf{B})^{/} \\
&\chi_*(\mathbf{AB})^{b} = \chi_*(\mathbf{A})^{/}\,\chi_*(\mathbf{B})^{b} = \chi_*(\mathbf{A})^{b}\,\chi_*(\mathbf{B})^{\backslash} \\
&\vdots \\
&\chi_*^{-1}(\mathbf{ab})^{/} = \chi_*^{-1}(\mathbf{a})^{b}\,\chi_*^{-1}(\mathbf{b})^{\#} = \chi_*^{-1}(\mathbf{a})^{/}\,\chi_*^{-1}(\mathbf{b})^{/} \\
&\chi_*^{-1}(\mathbf{ab})^{b} = \chi_*^{-1}(\mathbf{a})^{b}\,\chi_*^{-1}(\mathbf{b})^{\#} = \chi_*^{-1}(\mathbf{a})^{/}\,\chi_*^{-1}(\mathbf{b})^{/}
\end{aligned}
\qquad (2.12.51)
$$

Push-Forward and Pull-Back operations for Strain Tensors

The push-forward of the covariant right Cauchy-Green strain and its contravariant inverse are

$$
\begin{aligned}
\chi_*(\mathbf{C})^{b} &= C_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{F}^{-\mathrm{T}}\mathbf{C}\mathbf{F}^{-1} \\
\chi_*\left(\mathbf{C}^{-1}\right)^{\#} &= \left(C^{-1}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{F}\mathbf{C}\mathbf{F}^{\mathrm{T}}
\end{aligned}
. \qquad (2.12.52)
$$

From 2.10.39, $C_{ij} = g_{ij}$, the covariant components of the identity tensor expressed in terms of the convected base vectors in the current configuration, i.e. the spatial metric tensor, $\mathbf{g} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^i$, and $\left(C^{-1}\right)^{ij} = g^{ij}$, the contravariant components of **g**. Thus the push-forward of covariant **C** is **g** and the pull-back of covariant **g** is **C**, and the push-forward of contravariant $\mathbf{C}^{-1}$ is **g** and the pull-back of contravariant **g** is $\mathbf{C}^{-1}$:

$$
\boxed{
\begin{aligned}
\chi_*(\mathbf{C})^{b} = \mathbf{g}, \quad & \chi_*^{-1}(\mathbf{g})^{b} = \mathbf{C} \\
\chi_*\left(\mathbf{C}^{-1}\right)^{\#} = \mathbf{g}, \quad & \chi_*^{-1}(\mathbf{g})^{\#} = \mathbf{C}^{-1}
\end{aligned}
}
\qquad (2.12.53)
$$

Push-forward of the right Cauchy-Green strain

Similarly, the pull-back of covariant $\mathbf{b}^{-1}$ is **G** and the push-forward of covariant **G** is $\mathbf{b}^{-1}$**,** and the pull-back of contravariant **b** is **G** and the push-forward of contravariant **G** is **b**.

$$
\boxed{
\begin{aligned}
\chi_*(\mathbf{G})^{b} = \mathbf{b}^{-1}, \quad & \chi_*^{-1}\left(\mathbf{b}^{-1}\right)^{b} = \mathbf{G} \\
\chi_*(\mathbf{G})^{\#} = \mathbf{b}, \quad & \chi_*^{-1}(\mathbf{b})^{\#} = \mathbf{G}
\end{aligned}
}
\qquad (2.12.54)
$$

Pull-back of the left Cauchy-Green strain

For the covariant form of the Green-Lagrange strain, the push-forward is

$$
\chi_*(\mathbf{E})^{b} = E_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}. \qquad (2.12.55)
$$

From 2.10.43, $E_{ij} = e_{ij}$, the covariant components of the Euler-Almansi strain tensor, and so the push-forward of covariant **E** is **e** and the pull-back of covariant **e** is **E**.

$$\boxed{\chi_*(\mathbf{E})^b = \mathbf{e}, \qquad \chi_*^{-1}(\mathbf{e})^b = \mathbf{E}}. \tag{2.12.56}$$

Push-forward of the Green-Lagrange strain
Pull-back of the Euler-Almansi strain

Push-Forward and Pull-Back with Polar Decomposition Intermediate Configurations

Pull backs and push-forwards can be defined relative to any two configurations. Consider the polar decomposition and the intermediate configurations discussed in §2.10 (see Fig. 2.10.11). Effectively, we are replacing **F** with **R**: pushing forward a material tensor **A** from the reference configuration $\{\mathbf{G}_i\}$ to the configuration $\{\hat{\mathbf{G}}_i\}$ leads to

$$\begin{aligned}
\chi_*(\mathbf{A})^b{}_{\mathbf{R}(\mathbf{G})} &= A_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j = A_{ij}\left(\mathbf{R}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{R}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{R}^{-\mathrm{T}}\mathbf{A}\mathbf{R}^{-1} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^\#{}_{\mathbf{R}(\mathbf{G})} &= A^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j = A^{ij}\left(\mathbf{R}\mathbf{G}_i \otimes \mathbf{R}\mathbf{G}_j\right) = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^\backslash{}_{\mathbf{R}(\mathbf{G})} &= A^i_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j = A^i_{\cdot j}\left(\mathbf{R}\mathbf{G}_i \otimes \mathbf{R}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{R}\mathbf{A}\mathbf{R}^{-1} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^/{}_{\mathbf{R}(\mathbf{G})} &= A_i^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j = A_i^{\cdot j}\left(\mathbf{R}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{R}\mathbf{G}_j\right) = \mathbf{R}^{-\mathrm{T}}\mathbf{A}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}}
\end{aligned} . \tag{2.12.57}$$

Note that the result is the same regardless of whether one is using the covariant, contravariant or mixed forms.

Similarly, the pull back of a tensor $\hat{\mathbf{A}}$ from the intermediate configuration $\{\hat{\mathbf{G}}_i\}$ to the reference configuration $\{\mathbf{G}_i\}$ is

$$\begin{aligned}
\chi_*^{-1}(\hat{\mathbf{A}})^b{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^\#{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^\backslash{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^/{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R}
\end{aligned} \tag{2.12.58}$$

The push-forward of a tensor $\hat{\mathbf{a}}$ from $\{\hat{\mathbf{g}}_i\}$ to $\{\mathbf{g}\}$ and the corresponding pull-back of a spatial tensor **a** is

$$\begin{aligned}
\chi_*(\hat{\mathbf{a}})^b{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^b{}_{\mathbf{R}(\mathbf{g})} &= a_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^\#{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^\#{}_{\mathbf{R}(\mathbf{g})} &= a^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^\backslash{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}}, & \chi_*^{-1}(\mathbf{a})^\backslash{}_{\mathbf{R}(\mathbf{g})} &= a^i_{\cdot j}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^/{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^/{}_{\mathbf{R}(\mathbf{g})} &= a_i^{\cdot j}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R}
\end{aligned} \tag{2.12.59}$$

The push-forwards and pull-backs due to the stretch tensors are

$$\begin{aligned}
\chi_*(\mathbf{A})^{b}{}_{\mathbf{U}(\mathbf{G})} &= A_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j = A_{ij}\left(\mathbf{U}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{U}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{U}^{-\mathrm{T}}\mathbf{A}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}^{-1} \\
\chi_*(\mathbf{A})^{\#}{}_{\mathbf{U}(\mathbf{G})} &= A^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = A^{ij}\left(\mathbf{U}\mathbf{G}_i \otimes \mathbf{U}\mathbf{G}_j\right) = \mathbf{U}\mathbf{A}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\mathbf{A}\mathbf{U} \\
\chi_*(\mathbf{A})^{\backslash}{}_{\mathbf{U}(\mathbf{G})} &= A^{i}_{\cdot j}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j = A^{i}_{\cdot j}\left(\mathbf{U}\mathbf{G}_i \otimes \mathbf{U}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} \\
\chi_*(\mathbf{A})^{/}{}_{\mathbf{U}(\mathbf{G})} &= A_{i}^{\cdot j}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j = A_{i}^{\cdot j}\left(\mathbf{U}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{U}\mathbf{G}_j\right) = \mathbf{U}^{-\mathrm{T}}\mathbf{A}\mathbf{U}^{\mathrm{T}} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}
\end{aligned} \quad . \qquad (2.12.60)$$

$$\begin{aligned}
\chi_*^{-1}(\hat{\mathbf{a}})^{b}{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \mathbf{U}\hat{\mathbf{a}}\mathbf{U} \\
\chi_*^{-1}(\hat{\mathbf{a}})^{\#}{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \mathbf{U}^{-1}\hat{\mathbf{a}}\mathbf{U}^{-1} \\
\chi_*^{-1}(\hat{\mathbf{a}})^{\backslash}{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}^{i}_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = \mathbf{U}^{-1}\hat{\mathbf{a}}\mathbf{U} \\
\chi_*^{-1}(\hat{\mathbf{a}})^{/}{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}_{i}^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \mathbf{U}\hat{\mathbf{a}}\mathbf{U}^{-1}
\end{aligned} \qquad (2.12.61)$$

and

$$\begin{aligned}
\chi_*(\hat{\mathbf{A}})^{b}{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{v}^{-1}\hat{\mathbf{A}}\mathbf{v}^{-1} &\quad \chi_*^{-1}(\mathbf{a})^{b}{}_{\mathbf{v}(\mathbf{g})} &= a_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j = \mathbf{v}\mathbf{a}\mathbf{v} \\
\chi_*(\hat{\mathbf{A}})^{\#}{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{v}\hat{\mathbf{A}}\mathbf{v} &\quad \chi_*^{-1}(\mathbf{a})^{\#}{}_{\mathbf{v}(\mathbf{g})} &= a^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j = \mathbf{v}^{-1}\mathbf{a}\mathbf{v}^{-1} \\
\chi_*(\hat{\mathbf{A}})^{\backslash}{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}^{i}_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{v}\hat{\mathbf{A}}\mathbf{v}^{-1} &, \quad \chi_*^{-1}(\mathbf{a})^{\backslash}{}_{\mathbf{v}(\mathbf{g})} &= a^{i}_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j = \mathbf{v}^{-1}\mathbf{a}\mathbf{v} \\
\chi_*(\hat{\mathbf{A}})^{/}{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}_{i}^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \mathbf{v}^{-1}\hat{\mathbf{A}}\mathbf{v} &\quad \chi_*^{-1}(\mathbf{a})^{/}{}_{\mathbf{v}(\mathbf{g})} &= a_{i}^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j = \mathbf{v}\mathbf{a}\mathbf{v}^{-1}
\end{aligned} \qquad (2.12.62)$$

Push-forwards and pull-backs can also be defined using $\mathbf{F}^{\mathrm{T}}$ (in the place of $\mathbf{F}$) and these move between the intermediate configurations, $\hat{\mathbf{G}} \Leftrightarrow \hat{\mathbf{g}}$.

Recall Eqn. 2.10.64, which state that the covariant components of $\mathbf{U}, \mathbf{v}, \mathbf{U}^{-1}, \mathbf{v}^{-1}$ with respect to the bases $\mathbf{G}^i, \hat{\mathbf{G}}^i, \hat{\mathbf{g}}^i, \mathbf{g}^i$ respectively, are equal. This can be explained also in terms of push-forwards and pull-backs. For example, with $\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}$ and $\mathbf{v}^{-1} = \mathbf{R}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}}$, one can write (in fact these relations are valid for all component types)

$$\mathbf{v} = \chi_*(\mathbf{U})_{\mathbf{R}(\mathbf{G})}, \qquad \mathbf{v}^{-1} = \chi_*\left(\mathbf{U}^{-1}\right)_{\mathbf{R}(\hat{\mathbf{g}})} \qquad (2.12.63)$$

The first of these shows that the components of $\mathbf{U}$ with respect to $\mathbf{G}$ are the same as those of $\mathbf{v}$ with respect to $\hat{\mathbf{G}}$ (for all component types). The second shows that the components of $\mathbf{U}^{-1}$ with respect to $\hat{\mathbf{g}}$ are the same as those of $\mathbf{v}^{-1}$ with respect to $\mathbf{g}$.

As another example, with $\mathbf{C} = \mathbf{U}^2$,

$$\mathbf{C} = \chi_*^{-1}(\hat{\mathbf{g}})^b{}_{\mathbf{U}(\hat{\mathbf{g}})}, \qquad \mathbf{C}^{-1} = \chi_*^{-1}(\hat{\mathbf{g}})^\#{}_{\mathbf{U}(\hat{\mathbf{g}})} \tag{2.12.64}$$

2.12.3 The Lie Time Derivative

The **Lie (time) derivative** is a concept of tensor analysis which is used to distinguish between the change in some quantity, and the change in that quantity excluding changes due to the motion/configuration changes. As mentioned in the introduction to this section, we can imagine a hypothetical observer attached to the deforming material, who moves and deforms with the material. This observer will see no change in the configuration itself, $\dot{\mathbf{g}}_i = \dot{\mathbf{g}}^i = 0$. However, they will still see changes to vectors and tensors. These changes are measured using the Lie Derivative, which will be seen to be none other than the convected derivative discussed above.

Vectors

First, the **Lie (time) derivative** $\mathrm{L}_v \mathbf{v}$ of a vector $\mathbf{v}$ is the material derivative *holding the deformed basis constant*, that is, Eqns. 2.12.3:

$$\begin{aligned} \mathrm{L}_v^b\, \mathbf{v} &= \dot{v}_i \mathbf{g}^i \\ \mathrm{L}_v^\#\, \mathbf{v} &= \dot{v}^i \mathbf{g}_i \end{aligned} \tag{2.12.65}$$

Formally, it is defined in terms of the pull-back and push-forward,

$$\boxed{\mathrm{L}_v \mathbf{v} = \chi_*\left(\frac{d}{dt}\left[\chi_*^{-1}(\mathbf{v})\right]\right)} \qquad \textbf{The Lie Time Derivative} \tag{2.12.66}$$

This is illustrated in the Fig. 2.12.3. The spatial vector is first pulled back to the reference configuration, there the differentiation is carried out, where the base vectors are constant, then the vector is pushed forward again to the spatial description.

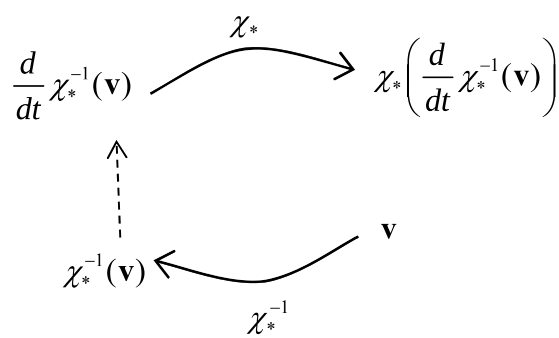


Figure 2.12.3: The Lie Derivative

For covariant components, one first pulls back the vector $v_i \mathbf{g}^i$ to $v_i \mathbf{G}^i$, the derivative is taken, $\dot{v}_i \mathbf{G}^i$, and then it is pushed forward to $\dot{v}_i \mathbf{g}^i$, which is consistent with the definition 2.12.65a. The definition 2.12.51 allows one to calculate the Lie derivative in absolute notation: using 2.12.36a, 2.12.33a, 2.11.9,

$$\begin{aligned} \mathrm{L}_v^b \mathbf{v} &= \chi_* \left(\frac{d}{dt} \left[\chi_*^{-1} \left(\mathbf{v} \right)^b \right] \right)^b = \mathbf{F}^{-\mathrm{T}} \left(\frac{d}{dt} \left[\mathbf{F}^{\mathrm{T}} \mathbf{v} \right] \right) \\ &= \mathbf{F}^{-\mathrm{T}} \left(\dot{\mathbf{F}}^{\mathrm{T}} \mathbf{v} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{v}} \right) \\ &= \mathbf{F}^{-\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}} \mathbf{l}^{\mathrm{T}} \mathbf{v} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{v}} \right) \\ &= \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v} \end{aligned} \qquad (2.12.67)$$

The Lie derivative for the contravariant components can be calculated in a similar way, and in summary (these are simply Eqns. 2.12.6): {▲Problem 2}

$$\boxed{\begin{aligned} \mathrm{L}_v^b \mathbf{v} &= \dot{v}_i \mathbf{g}^i = \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v} \\ \mathrm{L}_v^\# \mathbf{v} &= \dot{v}^i \mathbf{g}_i = \dot{\mathbf{v}} - \mathbf{l} \mathbf{v} \end{aligned}} \qquad \textbf{Lie Derivatives of Vectors} \qquad (2.12.68)$$

Tensors

The material time derivative of a spatial tensor **a** is

$$\begin{aligned} \dot{\mathbf{a}} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j + a_{ij} \dot{\mathbf{g}}^i \otimes \mathbf{g}^j + a_{ij} \mathbf{g}^i \otimes \dot{\mathbf{g}}^j \\ &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j + a^{ij} \dot{\mathbf{g}}_i \otimes \mathbf{g}_j + a^{ij} \mathbf{g}_i \otimes \dot{\mathbf{g}}_j \\ &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j + a^i_{\cdot j} \dot{\mathbf{g}}_i \otimes \mathbf{g}^j + a^i_{\cdot j} \mathbf{g}_i \otimes \dot{\mathbf{g}}^j \\ &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j + a_i^{\cdot j} \dot{\mathbf{g}}^i \otimes \mathbf{g}_j + a_i^{\cdot j} \mathbf{g}^i \otimes \dot{\mathbf{g}}_j \end{aligned} \qquad (2.12.69)$$

The Lie (time) derivative $\mathrm{L}_v \mathbf{a}$ is then

$$\begin{aligned} \mathrm{L}_v^b \mathbf{a} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \\ \mathrm{L}_v^\# \mathbf{a} &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \\ \mathrm{L}_v^\backslash \mathbf{a} &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j \\ \mathrm{L}_v^/ \mathbf{a} &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j \end{aligned} \qquad (2.12.70)$$

For example, for covariant components, one first pulls back the tensor $a_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$ to $a_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$, the derivative is taken, $\dot{a}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$, and then it is pushed forward to $\dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$. Thus, using 2.12.42a, 2.12.42a, 2.11.9,

$$
\begin{aligned}
\mathrm{L}_v^b \mathbf{a} &= \chi_* \left(\frac{d}{dt} \left[\chi_*^{-1} (\mathbf{a})^b \right] \right)^b \\
&= \mathbf{F}^{-\mathrm{T}} \left(\dot{\mathbf{F}}^{\mathrm{T}} \mathbf{a} \mathbf{F} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{a}} \mathbf{F} + \mathbf{F}^{\mathrm{T}} \mathbf{a} \dot{\mathbf{F}} \right) \mathbf{F}^{-1} \\
&= \mathbf{F}^{-\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}} \mathbf{l}^{\mathrm{T}} \mathbf{a} \mathbf{F} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{a}} \mathbf{F} + \mathbf{F}^{\mathrm{T}} \mathbf{a} \mathbf{l} \mathbf{F} \right) \mathbf{F}^{-1} \\
&= \mathbf{l}^{\mathrm{T}} \mathbf{a} + \dot{\mathbf{a}} + \mathbf{a} \mathbf{l}
\end{aligned}
\qquad (2.12.71)
$$

The Lie derivative for the other components can be calculated in a similar way, and in summary (these are Eqns. 2.12.14): {▲Problem 3}

$$
\boxed{
\begin{aligned}
\mathrm{L}_v^b \mathbf{a} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = \dot{\mathbf{a}} + \mathbf{l}^{\mathrm{T}} \mathbf{a} + \mathbf{a} \mathbf{l} \\
\mathrm{L}_v^\# \mathbf{a} &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = \dot{\mathbf{a}} - \mathbf{l} \mathbf{a} - \mathbf{a} \mathbf{l}^{\mathrm{T}} \\
\mathrm{L}_v^\backslash \mathbf{a} &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j = \dot{\mathbf{a}} - \mathbf{l} \mathbf{a} + \mathbf{a} \mathbf{l} \\
\mathrm{L}_v^/ \mathbf{a} &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j = \dot{\mathbf{a}} + \mathbf{l}^{\mathrm{T}} \mathbf{a} - \mathbf{a} \mathbf{l}^{\mathrm{T}}
\end{aligned}
}
\qquad \textbf{Lie Derivatives of Tensors} \qquad (2.12.72)
$$

The first of these, $\dot{\mathbf{a}} + \mathbf{l}^{\mathrm{T}} \mathbf{a} + \mathbf{a} \mathbf{l}$, is called the **Cotter-Rivlin rate**. The second of these, $\dot{\mathbf{a}} - \mathbf{l} \mathbf{a} - \mathbf{a} \mathbf{l}^{\mathrm{T}}$, is also called the **Oldroyd rate**.

Lie Derivatives of Strain Tensors

From 2.5.18,

$$
\begin{aligned}
\mathbf{d} &= \dot{\mathbf{e}} + \mathbf{l}^{\mathrm{T}} \mathbf{e} + \mathbf{e} \mathbf{l} \\
\dot{\mathbf{b}} - \mathbf{l} \mathbf{b} - \mathbf{b} \mathbf{l}^{\mathrm{T}} &= \mathbf{0}
\end{aligned}
\qquad (2.12.73)
$$

and so the Lie derivative of the covariant form of the Euler-Almansi strain is the rate of deformation and the Lie derivative of the contravariant form of the left Cauchy-Green tensor is zero. Further, from 2.12.53a, the Lie derivative of the metric tensor is the push forward of the material time derivative of the right Cauchy-Green strain:

$$
\mathrm{L}_v^b \mathbf{g} = \chi_* \left(\dot{\mathbf{C}} \right)^b, \qquad (2.12.74)
$$

Also, directly from 2.11.15,

$$
\mathrm{L}_v^b \mathbf{g} = 2 \mathbf{d} \qquad (2.12.75)
$$

Corotational Rates

The Lie derivatives in 2.12.72 were derived using pull-backs and push-forwards between the reference configuration and the current configuration. If, instead, we relate quantities to the rotated intermediate configuration, in other words use **R** instead of **F** in the calculations, we find that, using Eqn. 2.6.1, $\mathbf{\Omega_R} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$,

$$\begin{aligned} \mathrm{L}_v \mathbf{a} &= \chi_* \left(\frac{d}{dt} \left[\chi_*^{-1}(\mathbf{a}) \right] \right) \\ &= \mathbf{R} \left(\frac{d}{dt} \left[\mathbf{R}^{\mathrm{T}} \mathbf{a} \mathbf{R} \right] \right) \mathbf{R}^{\mathrm{T}} \\ &= \dot{\mathbf{a}} - \mathbf{\Omega_R} \mathbf{a} + \mathbf{a} \mathbf{\Omega_R} \end{aligned} \tag{2.12.76}$$

This is called the **Green-Naghdi rate**.

Rather than pulling back from the intermediate configuration to the reference configuration, we can choose the current configuration to be the reference configuration. Rotating from this configuration (see section 2.6.3), $\mathbf{\Omega_R} = \mathbf{w}$, the spin tensor, and one obtains the **Jaumann rate**, $\dot{\mathbf{a}} - \mathbf{wa} + \mathbf{aw}$.

Lie Derivatives and Objective Rates

The concept of objectivity was discussed in section 2.8. Essentially, if two observers are rotating relative to each other with rotation $\mathbf{Q}(t)$ and both are observing some spatial tensor, **T** as measured by one observer and $\mathbf{T}^*$ as measured by the other, then this tensor is objective provided $\mathbf{T}^* = \mathbf{QTQ}^{\mathrm{T}}$ for all **Q**, i.e. the measurement of the deformation would be independent of the observer. One of the most important uses of the Lie derivative is that *Lie derivatives of objective spatial tensors are objective spatial tensors*. Thus the rates given in 2.12.72 are all objective.

For example, suppose we have an objective spatial tensor **a**, i.e. so that $\mathbf{a}^* = \mathbf{QaQ}^{\mathrm{T}}$. The velocity gradient is not objective, and instead satisfies the relation 2.8.27: $\mathbf{l}^* = \mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. Using the properties of the transpose, the orthogonality of **Q**, and the identity $\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}$, one has for Eqns. 2.12.72a,b,

$$
\begin{aligned}
\left(\mathrm{L}_v^b \mathbf{a}\right)^* &= \overline{\mathbf{a}^*}^{\cdot} + \mathbf{l}^{*\mathrm{T}}\mathbf{a}^* + \mathbf{a}^*\mathbf{l}^* \\
&= \overline{\mathbf{QaQ}^{\mathrm{T}}}^{\cdot} + \left(\mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{QaQ}^{\mathrm{T}}\right) + \left(\mathbf{QaQ}^{\mathrm{T}}\right)\left(\mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right) \\
&= \dot{\mathbf{Q}}\mathbf{aQ}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{a}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Qa}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Ql}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{QaQ}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{QaQ}^{\mathrm{T}} \\
&\quad + \mathbf{QaQ}^{\mathrm{T}}\mathbf{QlQ}^{\mathrm{T}} + \mathbf{QaQ}^{\mathrm{T}}\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} \\
&= \mathbf{Q}\left(\dot{\mathbf{a}} + \mathbf{l}^{\mathrm{T}}\mathbf{a} + \mathbf{al}\right)\mathbf{Q}^{\mathrm{T}} \\
\left(\mathrm{L}_v^\# \mathbf{a}\right)^* &= \overline{\mathbf{a}^*}^{\cdot} - \mathbf{l}^*\mathbf{a}^* - \mathbf{a}^*\mathbf{l}^{*\mathrm{T}} \\
&= \overline{\mathbf{QaQ}^{\mathrm{T}}}^{\cdot} - \left(\mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right)\left(\mathbf{QaQ}^{\mathrm{T}}\right) - \left(\mathbf{QaQ}^{\mathrm{T}}\right)\left(\mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}} \\
&= \dot{\mathbf{Q}}\mathbf{aQ}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{a}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Qa}\dot{\mathbf{Q}}^{\mathrm{T}} - \mathbf{QlQ}^{\mathrm{T}}\mathbf{QaQ}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\mathbf{QaQ}^{\mathrm{T}} \\
&\quad - \mathbf{QaQ}^{\mathrm{T}}\mathbf{Ql}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} - \mathbf{QaQ}^{\mathrm{T}}\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} \\
&= \mathbf{Q}\left(\dot{\mathbf{a}} - \mathbf{la} - \mathbf{al}^{\mathrm{T}}\right)\mathbf{Q}^{\mathrm{T}}
\end{aligned}
\tag{2.12.77}
$$

showing that these rates are indeed objective.

Further, any linear combination of them is objective, for example,

$$
\frac{1}{2}\left[\left(\dot{\mathbf{a}} + \mathbf{l}^{\mathrm{T}}\mathbf{a} + \mathbf{al}\right) + \left(\dot{\mathbf{a}} - \mathbf{la} - \mathbf{al}^{\mathrm{T}}\right)\right] = \dot{\mathbf{a}} + \frac{1}{2}\left[-\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right)\mathbf{a} + \mathbf{a}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right)\right] = \dot{\mathbf{a}} - \mathbf{wa} + \mathbf{aw} \tag{2.12.78}
$$

is objective, provided $\mathbf{a}$ is. This is the **Jaumann rate** introduced in Eqn. 2.8.36 and mentioned after Eqn. 2.12.76 above. Further, as mentioned after Eqn. 2.12.72, the **Cotter-Rivlin rate** of Eqn. 2.8.37 is equivalent to $\mathrm{L}_v^b \mathbf{a}$.

The Lie Derivative and the Directional Derivative

Recall that the material time derivative of a tensor can be written in terms of the directional derivative, §2.6.5. Hence the Lie derivative can also be expressed as

$$
\mathrm{L}_v \mathbf{T} = \chi_*\left(\partial_{\mathbf{f}}\left(\chi_*^{-1}(\mathbf{T})\right)[\mathbf{v}]\right) \tag{2.12.79}
$$

and hence the subscript v on the L. Thus one can say that the Lie derivative is the push forward of the directional derivative of the material field $\chi_*^{-1}(\mathbf{T})$ in the direction of the velocity vector.

2.12.4 Problems

1. Eqns. 2.12.30 follow immediately from 2.12.29. However, use Eqns. 2.12.40, 2.12.42, i.e. $\chi_*(\mathbf{A})^b = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{-1}$, etc., directly, to verify relations 2.12.45.
2. Derive the Lie derivatives of a vector **v**, Eqns. 2.12.68.
3. Derive the Lie derivatives of a tensor **a**, Eqns. 2.12.72.

2.13 Variation and Linearisation of Kinematic Tensors

2.13.1 The Variation of Kinematic Tensors

The Variation

In this section is reviewed the concept of the variation, introduced in Part I, §8.5.

The **variation** is defined as follows: consider a function $\mathbf{u}(\mathbf{x})$, with $\mathbf{u}^*(\mathbf{x})$ a second function which is at most infinitesimally different from $\mathbf{u}(\mathbf{x})$ at every point ***x***, Fig. 2.13.1

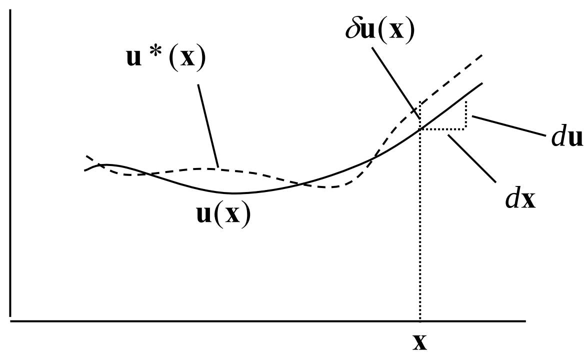


Figure 2.13.1: the variation

Then define

$$\boxed{\delta\mathbf{u} = \mathbf{u}^*(\mathbf{x}) - \mathbf{u}(\mathbf{x})} \qquad \textbf{The Variation} \qquad (2.13.1)$$

The operator δ is called the **variation symbol** and $\delta\mathbf{u}$ is called the variation of $\mathbf{u}(\mathbf{x})$.

The variation of $\mathbf{u}(\mathbf{x})$ is understood to represent an infinitesimal change in the function *at* ***x***. Note from the figure that a variation $\delta\mathbf{u}$ of a function **u** is different to a differential $d\mathbf{u}$. The ordinary differentiation gives a measure of the change of a function resulting from a specified change in the *independent* variable (in this case **x**). Also, note that the independent variable does not participate in the variation process; the variation operator imparts an infinitesimal change to the function **u** at some *fixed* **x** – formally, one can write this as $\delta\,\mathbf{x} = 0$.

The Commutative Properties of the variation operator

(1) $$\frac{d}{d\mathbf{x}}\delta\mathbf{u} = \delta\frac{d\mathbf{u}}{d\mathbf{x}} \qquad (2.13.2)$$

Proof:

$$\delta \frac{d\mathbf{u}}{d\mathbf{x}} = \left(\frac{d\mathbf{u}}{d\mathbf{x}}\right)^* - \frac{d\mathbf{u}}{d\mathbf{x}} = \frac{d\mathbf{u}^*}{d\mathbf{x}} - \frac{d\mathbf{u}}{d\mathbf{x}} = \frac{d(\mathbf{u}^* - \mathbf{u})}{d\mathbf{x}} = \frac{d}{d\mathbf{x}}\left(\delta \mathbf{u}(\mathbf{x})\right)$$

(2) $$\delta \int_{x_1}^{x_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{x_1}^{x_2} \delta \mathbf{u}(\mathbf{x}) d\mathbf{x} \qquad (2.13.3)$$

Proof:

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}^*(\mathbf{x}) d\mathbf{x} - \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} [\mathbf{u}^*(\mathbf{x}) - \mathbf{u}(\mathbf{x})] d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \delta \mathbf{u}(\mathbf{x}) d\mathbf{x}$$

Variation of a Function

Consider **A**, a scalar-, vector-, or tensor-valued function of $\mathbf{u}$, $\mathbf{A}(\mathbf{u})$. When we apply a variation to **u**, $\delta\mathbf{u}$, **A** changes to $\mathbf{A}(\mathbf{u} + \delta\mathbf{u})$. The variation of **A** is then defined as

$$\delta \mathbf{A}(\mathbf{u}, \delta\mathbf{u}) = \mathbf{A}(\mathbf{u} + \delta\mathbf{u}) - \mathbf{A}(\mathbf{u}) \qquad (2.13.4)$$

(in the limit as $\delta\mathbf{u} \to 0$). This can be expressed using the concept of the directional derivative in the usual way (see §1.6.11): consider the function $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})$, so that $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})_{\varepsilon=0} = \mathbf{A}(\mathbf{u})$ and $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})_{\varepsilon=1} = \mathbf{A}(\mathbf{u} + \delta\mathbf{u})$. A Taylor expansion gives $\mathbf{A}(\varepsilon) = \mathbf{A}(0) + \varepsilon(d\mathbf{A}/d\varepsilon)_{\varepsilon=0} + \cdots$, or

$$\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) = \mathbf{A}(\mathbf{u}) + \varepsilon\left(\frac{d}{d\varepsilon}\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})\right)_{\varepsilon=0} + \cdots \qquad (2.13.5)$$

Setting $\varepsilon = 1$ then gives Eqn. 2.13.4; thus

$$\mathbf{A}(\mathbf{u} + \delta\mathbf{u}) \approx \mathbf{A}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}] \qquad (2.13.6)$$

where $\partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}]$ is the directional derivative of **A** in the direction $\delta\mathbf{u}$; the directional derivative in this context is the variation of **A**:

$$\delta\mathbf{A}(\mathbf{u}, \delta\mathbf{u}) \equiv \partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) \qquad (2.13.7)$$

For example, consider the scalar function $\phi = \mathbf{P} : \mathbf{E}$, where **P** and **E** are second order tensors. Then

$$\delta\phi(\mathbf{E}, \delta\mathbf{E}) \equiv \partial_{\mathbf{E}}\phi[\delta\mathbf{E}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{P} : (\mathbf{E} + \varepsilon\delta\mathbf{E}) = \mathbf{P} : \delta\mathbf{E} \qquad (2.13.8)$$

The **second variation** is defined as

$$\delta^2 \mathbf{A} = \delta(\delta\mathbf{A}) = \partial_{\mathbf{u}}\delta\mathbf{A}[\delta\mathbf{u}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) \qquad (2.13.9)$$

For example, for a scalar function $\phi(\mathbf{u})$ of a vector **u**, the chain rule and Eqn. 2.13.2 give

$$\begin{aligned} \delta\varphi(\mathbf{u}, \delta\mathbf{u}) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \varphi(\mathbf{u} + \varepsilon\delta\mathbf{u}) = \left.\frac{d\varphi(\mathbf{u} + \varepsilon\delta\mathbf{u})}{d(\mathbf{u} + \varepsilon\delta\mathbf{u})}\right|_{\varepsilon=0} \frac{d(\mathbf{u} + \varepsilon\delta\mathbf{u})}{d\varepsilon} = \frac{\partial\varphi}{\partial\mathbf{u}} \cdot \delta\mathbf{u} \\ \delta^2\varphi &= \frac{\partial\delta\varphi}{\partial\mathbf{u}} \cdot \delta\mathbf{u} = \left(\delta\frac{\partial\varphi}{\partial\mathbf{u}}\right) \cdot \delta\mathbf{u} = \left(\frac{\partial^2\varphi}{\partial\mathbf{u}\partial\mathbf{u}}\delta\mathbf{u}\right) \cdot \delta\mathbf{u} = \delta\mathbf{u}\frac{\partial^2\varphi}{\partial\mathbf{u}\partial\mathbf{u}}\delta\mathbf{u} \end{aligned} \qquad (2.13.10)$$

Variation of Functions of the Displacement

In what follows is discussed the change (variation) in functions $\mathbf{A}(\mathbf{u})$ when the displacement (or velocity) fields undergo a variation. These ideas are useful in formulating variational principles of mechanics (see, for example, §3.9).

Shown in Fig. 2.13.2 is the current configuration frozen at some instant in time. The displacement field is then allowed to undergo a variation $\delta\mathbf{u}$. This change to the displacement field evidently changes kinematic tensors, and these changes are now investigated. Note that this variation to the displacement induces a variation to **x**, $\delta\mathbf{x}$, but **X** remains unchanged, $\delta\mathbf{X} = 0$.

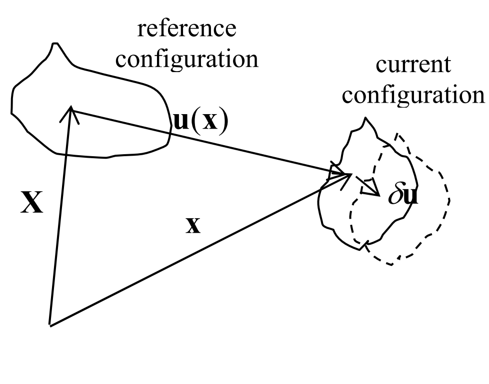


Figure 2.13.2: a variation of the displacement

To evaluate the variation of the deformation gradient $\mathbf{F}$, $\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})$, where $\mathbf{u}$ is the displacement field, note that $\mathbf{u}=\mathbf{x}-\mathbf{X}$ and Eqn. 2.2.43, $\mathbf{F}(\mathbf{u})=\text{Grad}\mathbf{u}+\mathbf{I}$. One has, from the definition 2.13.7,

$$\begin{aligned}\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})=\partial_{\mathbf{u}}\mathbf{F}[\delta\mathbf{u}]&=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})\\&=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left[\mathbf{F}(\mathbf{u})+\varepsilon\text{Grad}(\delta\mathbf{u})+\mathbf{I}\right]\\&=\text{Grad}(\delta\mathbf{u})\end{aligned}\tag{2.13.11}$$

Noting the first commutative property of the variation, 2.13.2, this can also be expressed as

$$\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})=\delta(\text{Grad}\mathbf{u})\tag{2.13.12}$$

Note that $\delta\mathbf{u}$ is completely independent of the function $\mathbf{u}$.

Here are some other examples, involving the inverse deformation gradient, the Green-Lagrange strain, the inverse right Cauchy-Green strain and the spatial line element: {▲Problem 1-3}

$$\begin{aligned}\delta\mathbf{F}^{-1}&=-\mathbf{F}^{-1}\text{grad}\delta\mathbf{u}\\\delta\mathbf{E}&=\mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\\\delta\mathbf{C}^{-1}&=-2\mathbf{F}^{-1}\boldsymbol{\varepsilon}\mathbf{F}^{-\mathrm{T}}\end{aligned}\tag{2.13.13}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor, Eqn. 2.2.48.

One also has, using the chain rule for the directional derivative, Eqn. 1.15.28, the directional derivative for the determinant, Eqn. 1.15.32, the trace relation 1.10.10e, Eqn. 2.2.8b,

$$\begin{aligned}\delta J(\mathbf{u},\delta\mathbf{u})&=\delta\det\mathbf{F}(\mathbf{u},\delta\mathbf{u})\\&=\partial_{\mathbf{u}}\det\mathbf{F}[\delta\mathbf{u}]\\&=\partial_{\mathbf{F}}\det\mathbf{F}[\partial_{\mathbf{u}}\mathbf{F}[\delta\mathbf{u}]]\\&=\partial_{\mathbf{F}}\det\mathbf{F}[\text{Grad}(\delta\mathbf{u})]\\&=\det\mathbf{F}\left[\mathbf{F}^{-\mathrm{T}}:\text{Grad}(\delta\mathbf{u})\right]\\&=J\text{tr}\left(\text{Grad}(\delta\mathbf{u})\mathbf{F}^{-1}\right)\\&=J\text{tr}(\text{grad}(\delta\mathbf{u}))\\&=J\,\text{div}(\delta\mathbf{u})\end{aligned}\tag{2.13.14}$$

Example

To put some of the above concepts into a simple and less abstract setting, consider the following scenario: a bar over $0 \le \mathbf{X} \le 1$ is extended, as illustrated in Fig. 2.13.3, according to:

$$\begin{aligned} \mathbf{x} &= 2\mathbf{X}^2 + 3 \\ \mathbf{X} &= \sqrt{\tfrac{1}{2}(\mathbf{x}-3)} \end{aligned} \tag{2.13.15}$$

The deformation gradient is

$$\mathbf{F} = \operatorname{Grad}\mathbf{x} = 4\mathbf{X} \tag{2.13.16}$$

So, for example, in the initial configuration (*A*), an infinitesimal line element at $\mathbf{X} = 0$ does not stretch ($\mathbf{F} = 0$) whereas a line element at $\mathbf{X} = 1$ stretches by 4.

The inverse deformation gradient is

$$\mathbf{F}^{-1} = \operatorname{grad}\mathbf{X} = \frac{1}{\sqrt{8(\mathbf{x}-3)}} \tag{2.13.17}$$

This implies that, in the current configuration (*B*), an infinitesimal line element at $\mathbf{x} = 3$ is the same size as its counterpart in the initial configuration ($\mathbf{F}^{-1} = 0$) whereas a line element at $\mathbf{x} = 5$ shrinks by a factor of 4 when returning to the initial configuration

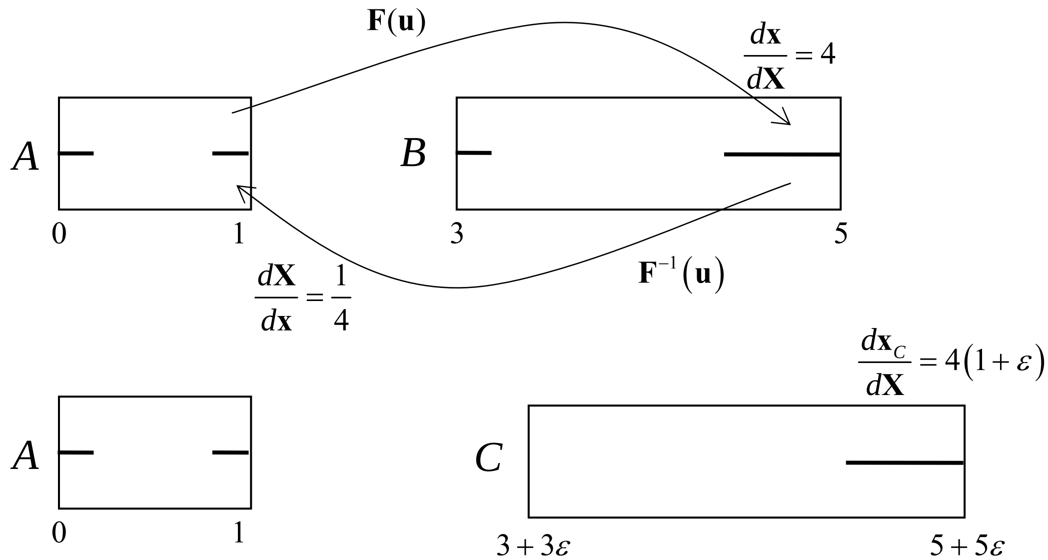


Figure 2.13.3: a motion and a variation

Now introduce a variation, which moves the bar from configuration *B* to configuration *C*:

$$\delta \mathbf{u} = \varepsilon \mathbf{x} = \varepsilon\left(2\mathbf{X}^2 + 3\right) \tag{2.13.18}$$

The point at 3 moves to $3 + 3\varepsilon$ and the point at 5 moves to $5 + 5\varepsilon$. (This variation happens to be a simple linear function of **x**, but it can be anything for our purposes here.)

The variation is plotted below as a function of **X** and **x**.

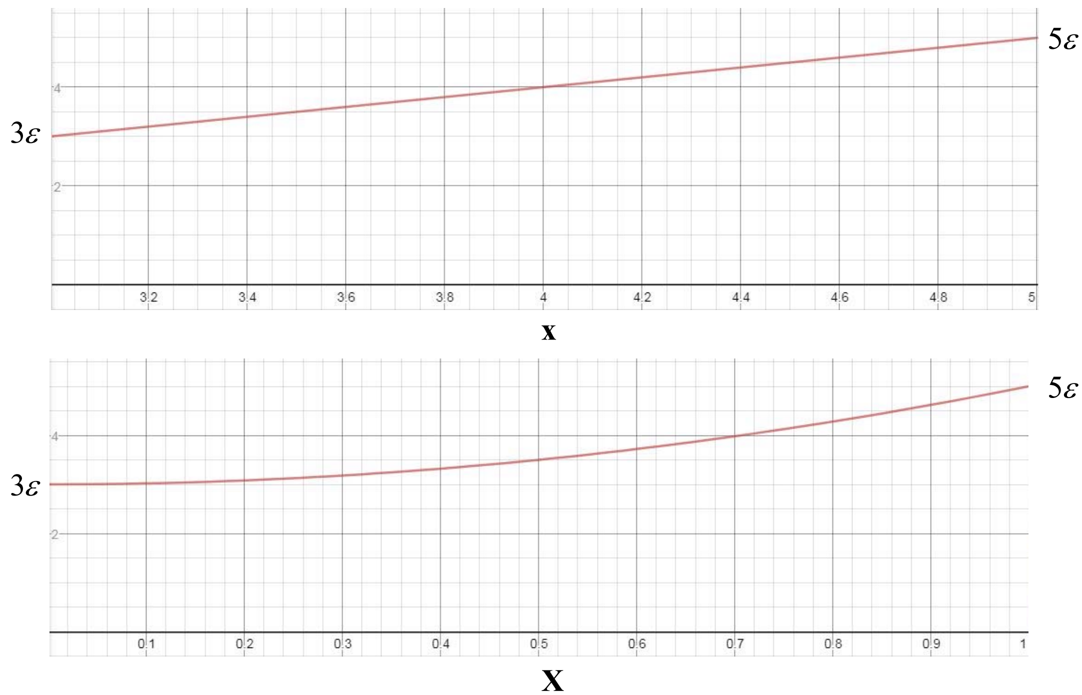


Figure 2.13.4: the variation as a function of x and X

Differentiating Eqns. 2.13.19, the gradients of the variations are

$$\begin{aligned} \operatorname{Grad}(\delta \mathbf{u}) &= \varepsilon(4\mathbf{X}) \\ \operatorname{grad}(\delta \mathbf{u}) &= \varepsilon \end{aligned} \tag{2.13.20}$$

which are the slopes in Figure 2.13.4.

To calculate the **F** associated with the new variation configuration, i.e. $\mathbf{F}(\mathbf{u} + \delta\mathbf{u})$, note that points **X** have now moved to:

$$2\mathbf{X}^2 + 3 + \varepsilon\left(2\mathbf{X}^2 + 3\right) \tag{2.13.21}$$

and so

$$\mathbf{F}(\mathbf{u}+\delta\mathbf{u}) = \mathrm{Grad}\left((1+\varepsilon)(2\mathbf{X}^2+3)\right) = 4\mathbf{X}+\varepsilon 4\mathbf{X} \tag{2.13.22}$$

This says that an infinitesimal line element at $\mathbf{X}=0$ does not stretch when moving to configuration *C* ($\mathbf{F}=0$) whereas a line element at $\mathbf{X}=1$ stretches by $4+4\varepsilon$.

Subtracting Eqn. 2.13.17 form Eqn. 2.13.22:

$$\delta\mathbf{F} = \mathbf{F}(\mathbf{u}+\delta\mathbf{u}) - \mathbf{F}(\mathbf{u}) = \varepsilon(4\mathbf{X}) \tag{2.13.23}$$

From Eqn. 2.13.20, this verifies Eqn. 2.13.11, that

$$\delta\mathbf{F} = \mathrm{Grad}(\delta\mathbf{u}) \tag{2.13.24}$$

We could also calculate the variation of **F** by moving directly from configuration *B* to configuration *C*. The movement of the particles from *B* to *C* is given by Eqn. 2.13.19: $\varepsilon(2\mathbf{X}^2+3)$ and so, based on this motion, $\delta\mathbf{F} = \mathrm{Grad}\left(\varepsilon(2\mathbf{X}^2+3)\right) = \varepsilon(4\mathbf{X})$.

To calculate the $\mathbf{F}^{-1}$ associated with the new variation configuration, i.e. $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u})$, note that the "new" current position **x** is (Eqn. 2.13.21):

$$\begin{aligned} \mathbf{x}_C &= 2\mathbf{X}^2+3+\varepsilon(2\mathbf{X}^2+3) \\ \rightarrow \mathbf{X} &= \sqrt{\frac{1}{2}\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)} \end{aligned} \tag{2.13.25}$$

This means that the point $3+3\varepsilon$ in configuration *C* corresponds to $\mathbf{X}=0$ and the point $5+5\varepsilon$ corresponds to $\mathbf{X}=1$. Then,

$$\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \frac{d}{d\mathbf{x}_C}\sqrt{\frac{1}{2}\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)} = \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)}} \tag{2.13.26}$$

So an element at the point $3+3\varepsilon$ in configuration *C* does not change in size as it is mapped back to the initial configuration, whereas an element at the point $5+5\varepsilon$ shrinks back to the initial configuration by a factor of $1/(4+4\varepsilon)$, as indicated in Fig. 2.13.3.

Alternatively, since $\mathbf{x}_C = \mathbf{x}+\varepsilon\mathbf{x}$, this can be written as

$$\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}} \tag{2.13.27}$$

Subtracting Eqn. 2.13.18 from Eqn. 2.13.27, the variation of the inverse deformation gradient is then

$$\begin{aligned}\delta\mathbf{F}^{-1}(\mathbf{u}) = \mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) - \mathbf{F}^{-1}(\mathbf{u}) &= \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}} - \frac{1}{\sqrt{8(\mathbf{x}-3)}} \\ &= -\frac{\varepsilon}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}}\end{aligned} \quad (2.13.28)$$

Using a series expansion, $(1+\varepsilon)^{-1} = 1-\varepsilon+\varepsilon^2-\ldots$, for small ε (neglecting terms of order ε^2),

$$\delta\mathbf{F}^{-1}(\mathbf{u}) = -\varepsilon\frac{1}{\sqrt{8(\mathbf{x}-3)}} \quad (2.13.29)$$

From Eqns. 2.13.18 and 2.13.20, this verifies the relation 2.13.13:

$$\delta\mathbf{F}^{-1}(\mathbf{u}) = -\mathbf{F}^{-1}\mathrm{grad}(\delta u) \quad (2.13.30)$$

A formula for the inverse deformation gradient is $\mathbf{F}^{-1} = \mathbf{I} - \mathrm{grad}\mathbf{u}$. However, note that $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) \neq \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{x}$, but that $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{x}_C$.

The Lie Variation

The **Lie-variation** is defined for *spatial* vectors and tensors as a variation holding the deformed basis constant. For example,

$$\delta_{\mathrm{L}}^{\mathrm{b}}\mathbf{a} = \delta a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \quad (2.13.31)$$

The object is first pulled-back, the variation is then taken and finally a push-forward is carried out. For example, analogous to 2.12.66,

$$\delta_{\mathrm{L}}\mathbf{a}(\mathbf{u},\delta\mathbf{u}) \equiv \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\mathbf{a})\right)[\delta\mathbf{u}]\right) \quad (2.13.32)$$

For example, consider the Lie-variation of the Euler-Almansi strain **e**. First, from 2.12.56b, $\chi_{-1}^*(\mathbf{e})^b = \mathbf{E}$. Then 2.13.13b gives $\partial_{\mathbf{u}}\left(\chi_{-1}^*(\mathbf{e})^b\right)[\delta\mathbf{u}] = \delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$. From 2.12.40a,

$$\delta_{\mathrm{L}}\mathbf{e}(\mathbf{u},\delta\mathbf{u}) = \chi_*\left(\partial_{\mathbf{u}}\left(\chi_{-1}^*(\mathbf{e})^b\right)[\delta\mathbf{u}]\right)^b = \chi_*\left(\mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\right)^b = \delta\boldsymbol{\varepsilon} \quad (2.13.33)$$

2.13.2 Linearisation of Kinematic Functions

Linearisation of a Function

As for the variation, consider **A**, a scalar-, vector-, or tensor-valued function of $\mathbf{u}$. If **u** undergoes an increment $\Delta\mathbf{u}$, then, analogous to 2.13.4,

$$\mathbf{A}(\mathbf{u}+\Delta\mathbf{u}) \approx \mathbf{A}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{A}[\Delta\mathbf{u}] \tag{1.13.34}$$

The directional derivative $\partial_{\mathbf{u}}\mathbf{A}[\Delta\mathbf{u}]$ in this context is also denoted by $\Delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u})$. The **linearization** of **A** with respect to **u** is defined to be

$$\mathrm{L}\ \mathbf{A}(\mathbf{u},\Delta\mathbf{u}) = \mathbf{A}(\mathbf{u}) + \Delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) \tag{1.13.35}$$

Using exactly the same method of calculation as was used for the variations above, the linearization of **F** and **E**, for example, are

$$\begin{aligned} \mathrm{L}\,\mathbf{F}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{F}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{F}[\Delta\mathbf{u}] = \mathbf{F} + \mathrm{Grad}\Delta\mathbf{u} \\ \mathrm{L}\,\mathbf{E}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{E}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{E}[\Delta\mathbf{u}] = \mathbf{E} + \mathbf{F}^{\mathrm{T}}\Delta\boldsymbol{\varepsilon}\mathbf{F} \end{aligned} \tag{2.13.36}$$

where $\Delta\boldsymbol{\varepsilon} = \frac{1}{2}\left((\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}} + (\mathrm{grad}\Delta\mathbf{u})\right)$ is the linearised small strain tensor $\boldsymbol{\varepsilon}$.

Linearisation of Variations of a Function

One can also linearise the variation of a function. For example,

$$\mathrm{L}\ \delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) = \delta\mathbf{A}(\mathbf{u},\delta\mathbf{u}) + \Delta\delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) \tag{2.13.37}$$

The second term here is the directional derivative

$$\begin{aligned} \Delta\delta\mathbf{A}[\mathbf{u},\Delta\mathbf{u}] &= \partial_{\mathbf{u}}\delta\mathbf{A}[\Delta\mathbf{u}] \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta\mathbf{A}(\mathbf{u}+\varepsilon\Delta\mathbf{u}) \end{aligned} \tag{2.13.38}$$

This leads to an expression similar to $\delta^2\mathbf{A}$. For example, for a scalar function $\phi(\mathbf{u})$ of a vector **u**,

$$\Delta\delta\phi = \frac{\partial\delta\phi}{\partial\mathbf{u}}\cdot\Delta\mathbf{u} = \Delta\mathbf{u}\frac{\partial^2\phi}{\partial\mathbf{u}\partial\mathbf{u}}\delta\mathbf{u} \tag{2.13.39}$$

Consider now the virtual Green-Lagrange strain, 2.13.11b, $\delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$. To carry out the linearization of $\delta\mathbf{E}$, it is convenient to first write it in the form

$$\begin{aligned}\delta\mathbf{E} &= \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F} \\ &= \tfrac{1}{2}\mathbf{F}^{\mathrm{T}}\left[(\mathrm{grad}\,\delta\mathbf{u})^{\mathrm{T}} + \mathrm{grad}\,\delta\mathbf{u}\right]\mathbf{F} \\ &= \tfrac{1}{2}\left[(\mathrm{Grad}\,\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\,\delta\mathbf{u}\right]\end{aligned} \tag{2.13.40}$$

Then

$$\Delta\delta\mathbf{E} = \partial_{\mathbf{u}}\delta\mathbf{E}[\Delta\mathbf{u}] = \partial_{\mathbf{u}}\left\{\tfrac{1}{2}\left[(\mathrm{Grad}\,\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\,\delta\mathbf{u}\right]\right\}[\Delta\mathbf{u}] \tag{2.13.41}$$

Recall that the variation $\delta\mathbf{u}$ is *independent* of $\mathbf{u}$; this equation is being linearised with respect to $\mathbf{u}$, and $\delta\mathbf{u}$ is unaffected by the linearization (see Fig. 2.13.3 below). However, the motion, and in particular $\mathbf{F}$, *are* affected by the increment in $\mathbf{u}$. Thus {▲Problem 4}

$$\Delta\delta\mathbf{E} = \mathrm{sym}\left((\mathrm{Grad}\,\Delta\mathbf{u})^{\mathrm{T}}\,\mathrm{Grad}\,\delta\mathbf{u}\right) \tag{2.13.42}$$

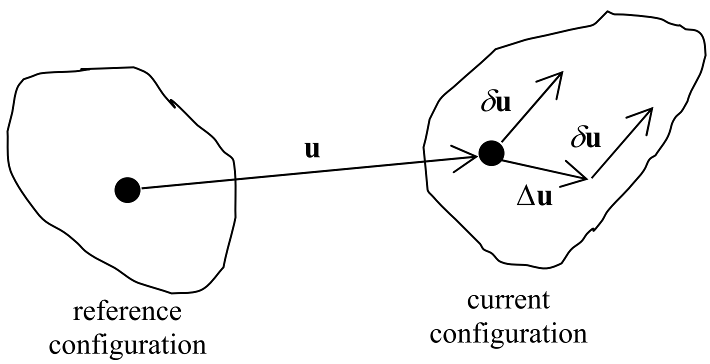


Figure 2.13.3: linearisation

As with the variational operator, one can define the linearization of a spatial tensor as involving a pull back, followed by the directional derivative, and finally the push forward operation. Thus

$$\Delta\mathbf{a}(\mathbf{u},\Delta\mathbf{u}) \equiv \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\mathbf{a})\right)[\Delta\mathbf{u}]\right) \tag{2.13.43}$$

2.13.3 Problems

1. Use Eqn. 2.2.22, $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right)$, Eqn. 2.13.11, $\delta\mathbf{F}(\mathbf{u}, \delta\mathbf{u}) = \mathrm{Grad}(\delta\mathbf{u})$, and Eqn. 2.2.8b, $\mathrm{grad}\mathbf{v} = (\mathrm{Grad}\mathbf{v})\mathbf{F}^{-1}$, to show that $\delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$, where $\boldsymbol{\varepsilon}$ is the small strain tensor, Eqn. 2.2.48.
2. Use 2.13.11 to show that the variation of the inverse deformation gradient $\mathbf{F}^{-1}$ is $\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathrm{grad}\delta\mathbf{u}$. [Hint: differente the relation $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$ by the product rule and then use the relation $\mathrm{grad}\mathbf{v} = (\mathrm{Grad}\mathbf{v})\mathbf{F}^{-1}$ for vector **v**.]
3. Use the definition $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ to show that $\delta\mathbf{C}^{-1} = -2\mathbf{F}^{-1}\boldsymbol{\varepsilon}\mathbf{F}^{-\mathrm{T}}$.
4. Use the relation $\mathrm{sym}\mathbf{A} = \frac{1}{2}\left(\mathbf{A}^{\mathrm{T}} + \mathbf{A}\right)$ to show that
$$\Delta\delta\mathbf{E} = \partial_{\mathbf{u}}\left\{\frac{1}{2}\left[(\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right]\right\}[\Delta\mathbf{u}] = \mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)$$
5. Use $\delta\mathbf{e} = \delta\boldsymbol{\varepsilon} = \frac{1}{2}\left[(\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}} + \mathrm{grad}\delta\mathbf{u}\right]$ to show that the
$$\begin{aligned}\Delta\delta\mathbf{e} &= \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\delta\mathbf{e})\right)[\Delta\mathbf{u}]\right) = \chi_*\mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)\\ &= \mathrm{sym}\left[(\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\cdot\mathrm{grad}\delta\mathbf{u}\right]\end{aligned}$$