Convected Coordinates: Time Rates of Change

In this section, the time derivatives of kinematic tensors described in §2.4-2.6 are now described using convected coordinates.

2.11.1 Deformation Rates

Time Derivatives of the Base Vectors and the Deformation Gradient

The material time derivatives of the material base vectors are zero: $\dot{\mathbf{G}}_i = \dot{\mathbf{G}}^i = 0$. The material time derivatives of the deformed base vectors are, from 2.10.23, (and using $\dot{\mathbf{I}} = d\left(\mathbf{F}\mathbf{F}^{-1}\right)/dt = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\dot{\mathbf{F}}^{-1}$)

$$\begin{aligned}\dot{\mathbf{g}}_i &= \dot{\mathbf{F}}\mathbf{G}_i = \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{g}_i = -\mathbf{F}\dot{\mathbf{F}}^{-1}\mathbf{g}_i \\ \dot{\mathbf{g}}^i &= \dot{\mathbf{F}}^{-\mathrm{T}}\mathbf{G}^i = \dot{\mathbf{F}}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{g}^i = -\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{g}^i\end{aligned} \tag{2.11.1}$$

with, again from 2.10.23,

$$\begin{aligned}\dot{\mathbf{F}} &= \dot{\mathbf{g}}_i \otimes \mathbf{G}^i \\ \dot{\mathbf{F}}^{-1} &= \mathbf{G}_i \otimes \dot{\mathbf{g}}^i \\ \dot{\mathbf{F}}^{-\mathrm{T}} &= \dot{\mathbf{g}}^i \otimes \mathbf{G}_i \\ \dot{\mathbf{F}}^{\mathrm{T}} &= \mathbf{G}^i \otimes \dot{\mathbf{g}}_i\end{aligned} \tag{2.11.2}$$

The Velocity Gradient

The velocity gradient is defined by 2.5.2, $\mathbf{l} = \operatorname{grad}\mathbf{v}$, so that, using 1.16.23,

$$\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial x^i} \otimes \mathbf{e}^i = \frac{\partial \mathbf{v}}{\partial \Theta^j}\frac{\partial \Theta^j}{\partial x^i} \otimes \mathbf{e}^i = \frac{\partial \mathbf{v}}{\partial \Theta^j} \otimes \mathbf{g}^j \tag{2.11.3}$$

Also, from 1.16.19,

$$\dot{\mathbf{g}}_i = \frac{\partial \dot{\mathbf{x}}}{\partial \Theta^i} = \frac{\partial \mathbf{v}}{\partial \Theta^i} \tag{2.11.4}$$

so that, as an alternative to 2.11.3,

$$\mathbf{l} = \dot{\mathbf{g}}_i \otimes \mathbf{g}^i \tag{2.11.5}$$

The components of the spatial velocity gradient are

$$
\begin{aligned}
l_{ij} &= \mathbf{g}_i \mathbf{l} \mathbf{g}_j = \mathbf{g}_i \cdot \dot{\mathbf{g}}_j \\
l^i_{\cdot j} &= \mathbf{g}^i \mathbf{l} \mathbf{g}_j = \mathbf{g}^i \cdot \dot{\mathbf{g}}_j \\
l_i^{\cdot j} &= \mathbf{g}_i \mathbf{l} \mathbf{g}^j = g^{mj} \mathbf{g}_i \cdot \dot{\mathbf{g}}_m = \mathbf{g}_i \cdot \dot{\mathbf{g}}^j \\
l^{ij} &= \mathbf{g}^i \mathbf{l} \mathbf{g}^j = \mathbf{g}^i \cdot \dot{\mathbf{g}}^j
\end{aligned}
\qquad (2.11.6)
$$

Convected Bases

From 2.11.1, 2.11.2 and 2.11.5,

$$
\begin{aligned}
\dot{\mathbf{g}}_i &= \mathbf{l} \mathbf{g}_i & \qquad \dot{\mathbf{g}}^i &= -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \\
&= \mathbf{g}_i \mathbf{l}^{\mathrm{T}} & &= -\mathbf{g}^i \mathbf{l}
\end{aligned}
\qquad (2.11.7)
$$

Contracting the first of these with $d\Theta^i$ leads to

$$
\dot{\mathbf{g}}_i d\Theta^i = \mathbf{l} \mathbf{g}_i d\Theta^i \qquad (2.11.8)
$$

which is equivalent to 2.5.1, $d\mathbf{v} = \mathbf{l} d\mathbf{x}$.

Time Derivatives of the Deformation Gradient in terms of the Velocity Gradient

Eqns. 2.11.2 can also be re-expressed using Eqns. 2.11.7:

$$
\begin{aligned}
\dot{\mathbf{F}} &= \dot{\mathbf{g}}_i \otimes \mathbf{G}^i = \mathbf{g}_i \mathbf{l}^{\mathrm{T}} \otimes \mathbf{G}^i = \mathbf{l} \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{l} \mathbf{F} \\
\dot{\mathbf{F}}^{-1} &= \mathbf{G}_i \otimes \dot{\mathbf{g}}^i = -\mathbf{G}_i \otimes \mathbf{g}^i \mathbf{l} = -\mathbf{F}^{-1} \mathbf{l} \\
\dot{\mathbf{F}}^{-\mathrm{T}} &= \dot{\mathbf{g}}^i \otimes \mathbf{G}_i = -\mathbf{g}^i \mathbf{l} \otimes \mathbf{G}_i = -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \otimes \mathbf{G}_i = -\mathbf{l}^{\mathrm{T}} \mathbf{F}^{-\mathrm{T}} \\
\dot{\mathbf{F}}^{\mathrm{T}} &= \mathbf{G}^i \otimes \dot{\mathbf{g}}_i = \mathbf{G}^i \otimes \mathbf{g}_i \mathbf{l}^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}} \mathbf{l}^{\mathrm{T}}
\end{aligned}
\qquad (2.11.9)
$$

which are Eqns. 2.5.4-5.

An alternative way of arriving at Eqns. 2.11.7 is to start with Eqns. 2.11.9: the covariant base vectors $\mathbf{G}_i$ convect to $\mathbf{g}_i(t)$ over time through the time-dependent deformation gradient: $\mathbf{g}_i(t) = \mathbf{F}(t)\mathbf{G}_i$. For this relation to hold at all times, one must have, from Eqn. 2.11.9b,

$$
\begin{aligned}
\dot{\mathbf{G}}_i = 0 &= \overline{\mathbf{F}^{-1} \mathbf{g}_i}^{\,\boldsymbol{\cdot}} \\
&= \dot{\mathbf{F}}^{-1} \mathbf{g}_i + \mathbf{F}^{-1} \dot{\mathbf{g}}_i \\
&= \mathbf{F}^{-1} \left(-\mathbf{l} \mathbf{g}_i + \dot{\mathbf{g}}_i \right)
\end{aligned}
\qquad (2.11.10)
$$

Thus, in order to maintain the convection of the tangent basis over time, one requires that

$$\dot{\mathbf{g}}_i = \mathbf{l}\mathbf{g}_i \tag{2.11.11}$$

The contravariant base vectors $\mathbf{G}^i$ transform to $\mathbf{g}^i(t)$ over time through the time-dependent inverse transpose of the deformation gradient: $\mathbf{g}^i(t) = \mathbf{F}^{-\mathrm{T}}(t)\mathbf{G}^i$. For this relation to hold at all times, one must have, from Eqn. 2.11.9d,

$$\begin{aligned} \dot{\mathbf{G}}^i = 0 &= \overline{\mathbf{F}^{\mathrm{T}}\mathbf{g}^i}^{\boldsymbol{\cdot}} \\ &= \dot{\mathbf{F}}^{\mathrm{T}}\mathbf{g}^i + \mathbf{F}^{\mathrm{T}}\dot{\mathbf{g}}^i \\ &= \mathbf{F}^{\mathrm{T}}\left(\mathbf{l}^{\mathrm{T}}\mathbf{g}^i + \dot{\mathbf{g}}^i\right) \end{aligned} \tag{2.11.12}$$

Thus, in order to maintain the convection of the normal basis over time, one requires that

$$\dot{\mathbf{g}}^i = -\mathbf{l}^{\mathrm{T}}\mathbf{g}^i \tag{2.11.13}$$

The Rate of Deformation and Spin Tensors

From 2.5.6, $\mathbf{l} = \mathbf{d} + \mathbf{w}$. The covariant components of the rate of deformation and spin are

$$\begin{aligned} d_{ij} &= \frac{1}{2}\mathbf{g}_i\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right)\mathbf{g}_j = \frac{1}{2}\mathbf{g}_i\left(\dot{\mathbf{g}}_m \otimes \mathbf{g}^m + \mathbf{g}^m \otimes \dot{\mathbf{g}}_m\right)\mathbf{g}_j = \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right) = \frac{1}{2}\overline{\mathbf{g}_i \cdot \mathbf{g}_j}^{\boldsymbol{\cdot}} \\ w_{ij} &= \frac{1}{2}\mathbf{g}_i\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right)\mathbf{g}_j = \frac{1}{2}\mathbf{g}_i\left(\dot{\mathbf{g}}_m \otimes \mathbf{g}^m - \mathbf{g}^m \otimes \dot{\mathbf{g}}_m\right)\mathbf{g}_j = \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j - \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right) \end{aligned} \tag{2.11.14}$$

Alternatively, from 2.11.6a,

$$\begin{aligned} \mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right) &= \frac{1}{2}\left(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \dot{\mathbf{g}}_i \cdot \mathbf{g}_j\right)\mathbf{g}_i \otimes \mathbf{g}_j \\ &= \frac{1}{2}\overline{\mathbf{g}_i \cdot \mathbf{g}_j}^{\boldsymbol{\cdot}}\,\mathbf{g}_i \otimes \mathbf{g}_j \\ &= \frac{1}{2}\dot{g}_{ij}\mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \tag{2.11.15}$$