2.9 Rigid Body Rotations of Configurations

In this section are discussed rigid body rotations to the current and reference configurations.

2.9.1 A Rigid Body Rotation of the Current Configuration

As mentioned in §2.8.1, the circumstance of two observers, moving relative to each other and examining a fixed configuration (the current configuration) is equivalent to one observer taking measurements of two different configurations, moving relative to each other[1]. The objectivity requirements of the various kinematic objects discussed in the previous section can thus also be examined by considering rigid body rotations and translations of the current configuration.

Any rigid body rotation and translation of the current configuration can be expressed in the form

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}(t)\mathbf{x}(\mathbf{X},t) + \mathbf{c}(t) \tag{2.9.1}$$

where $\mathbf{Q}$ is a rotation tensor. This is illustrated in Fig. 2.9.5. The current configuration is denoted by S and the rotated configuration by S^*.

Just as $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, the deformation gradient for the configuration S^* relative to the reference configuration S_0 is defined through $d\mathbf{x}^* = \mathbf{F}^* d\mathbf{X}$. From 2.9.1, as in §2.8.5 (see Eqn. 2.8.23), and similarly for the right and left Cauchy-Green tensors,

$$\begin{aligned} \mathbf{F}^* &= \mathbf{QF} \\ \mathbf{C}^* &= \mathbf{F}^{*\mathrm{T}}\mathbf{F}^* = \mathbf{C} \\ \mathbf{b}^* &= \mathbf{F}^*\mathbf{F}^{*\mathrm{T}} = \mathbf{QbQ}^{\mathrm{T}} \end{aligned} \tag{2.9.2}$$

Thus in the deformations $\mathbf{F}: S_0 \to S$ and $\mathbf{F}^*: S_0 \to S^*$, the right Cauchy Green tensors, $\mathbf{C}$ and $\mathbf{C}^*$, are the same, but the left Cauchy Green tensors are different, and related through $\mathbf{b}^* = \mathbf{QbQ}^{\mathrm{T}}$.

All the other results obtained in the last section in the context of observer transformations, for example for the Jacobian, stretch tensors, etc., hold also for the case of rotations to the current configuration.

[1] Although equivalent, there is a difference: in one, there are two observers who record one event (a material particle say) as at two different points, in the other there is one observer who records two different events (the place where the one material particle is in two different configurations)

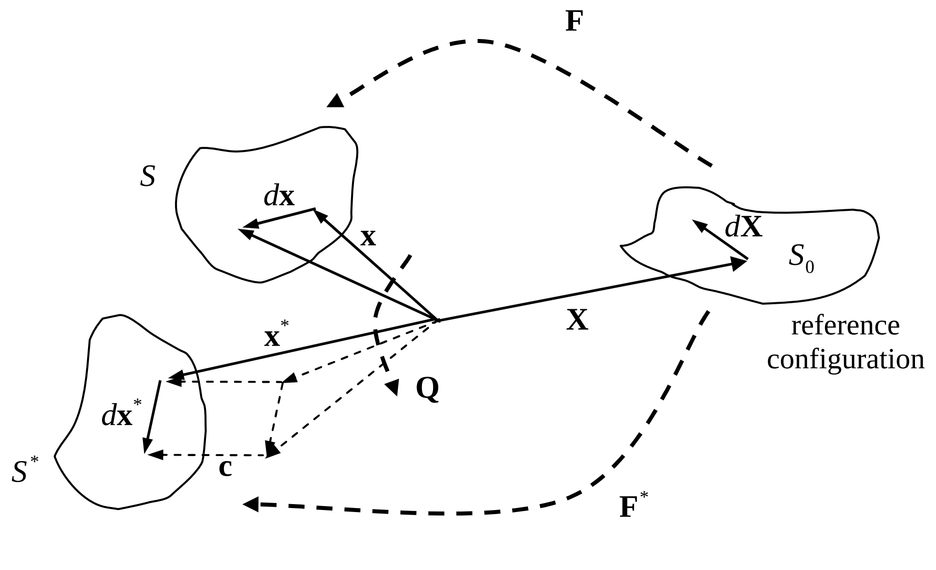

Figure 2.9.1: a rigid body rotation and translation of the current configuration

2.9.2 A Rigid Body Rotation of the Reference Configuration

Consider now a rigid-body rotation to the *reference* configuration. Such rotations play an important role in the notion of material symmetry (see Chapter 5).

The reference configuration is denoted by S_0 and the rotated/translated configuration by $S^\Diamond$, Fig. 2.9.2. The deformation gradient for the current configuration S relative to $S^\Diamond$ is defined through $d\mathbf{x} = \mathbf{F}^\Diamond d\mathbf{X}^\Diamond = \mathbf{F}^\Diamond \mathbf{Q} d\mathbf{X}$. But $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ and so (and similarly for the right and left Cauchy-Green tensors)

$$
\begin{aligned}
\mathbf{F}^\Diamond &= \mathbf{F}\mathbf{Q}^{\mathrm{T}} \\
\mathbf{C}^\Diamond &= \mathbf{F}^{\Diamond\mathrm{T}}\mathbf{F}^\Diamond = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}} \\
\mathbf{b}^\Diamond &= \mathbf{F}^\Diamond\mathbf{F}^{\Diamond\mathrm{T}} = \mathbf{b}
\end{aligned}
\tag{2.9.3}
$$

Thus the change to the right (left) Cauchy-Green strain tensor under a rotation to the reference configuration is the same as the change to the left (right) Cauchy-Green strain tensor under a rotation of the current configuration.

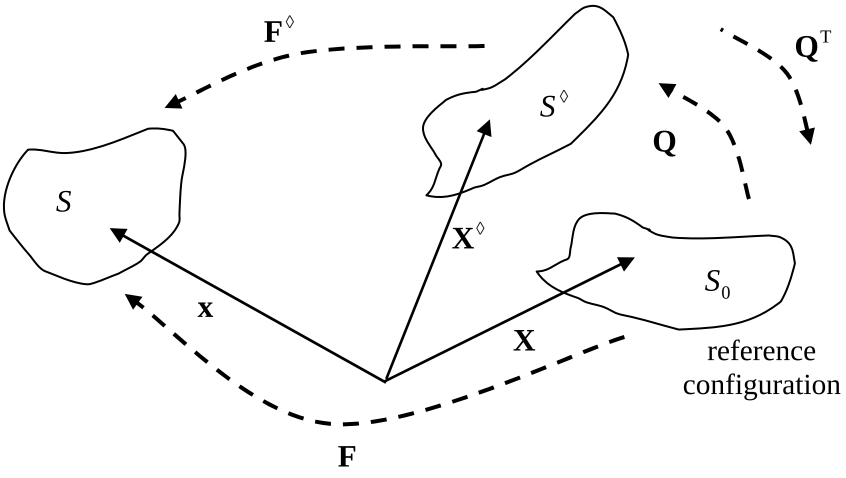

Figure 2.9.2: a rigid body rotation of the reference configuration