2.6 Deformation Rates: Further Topics

2.6.1 Relationship between l, d, w and the rate of change of R and U

Consider the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$. Since **R** is orthogonal, $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$, and a differentiation of this equation leads to

$$\boldsymbol{\Omega}_{\mathbf{R}} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \tag{2.6.1}$$

with $\boldsymbol{\Omega}_{\mathbf{R}}$ skew-symmetric (see Eqn. 1.14.2). Using this relation, the expression $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and the definitions of **d** and **w**, Eqn. 2.5.7, one finds that {▲Problem 1}

$$\begin{aligned} \mathbf{l} &= \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\ \mathbf{w} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\ &= \mathbf{R}\,\mathrm{skew}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\ \mathbf{d} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} \\ &= \mathbf{R}\,\mathrm{sym}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}} \end{aligned} \tag{2.6.2}$$

Note that $\boldsymbol{\Omega}_{\mathbf{R}}$ being skew-symmetric is consistent with **w** being skew-symmetric, and that both **w** and **d** involve **R**, and the rate of change of **U**.

When the motion is a rigid body rotation, then $\dot{\mathbf{U}} = \mathbf{0}$, and

$$\mathbf{w} = \boldsymbol{\Omega}_{\mathbf{R}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{2.6.3}$$

2.6.2 Deformation Rate Tensors and the Principal Material and Spatial Bases

The rate of change of the stretch tensor in terms of the principal material base vectors is

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\left\{\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \lambda_i\dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \lambda_i\hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i\right\} \tag{2.6.4}$$

Consider the case when the principal material axes stay constant, as can happen in some simple deformations. In that case, $\dot{\mathbf{U}}$ and $\mathbf{U}^{-1}$ are coaxial (see §1.11.5):

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \quad \text{and} \quad \mathbf{U}^{-1} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \tag{2.6.5}$$

with $\dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}}$ and, as expected, from 2.5.25b, $\mathbf{w} = \mathbf{\Omega_R} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, that is, any spin is due to rigid body rotation.

Similarly, from 2.2.37, and differentiating $\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \mathbf{I}$,

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \left\{ \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i \right\}. \tag{2.6.6}$$

Also, differentiating $\hat{\mathbf{N}}_i \cdot \hat{\mathbf{N}}_j = \delta_{ij}$ leads to $\dot{\hat{\mathbf{N}}}_i \cdot \hat{\mathbf{N}}_j = -\hat{\mathbf{N}}_i \cdot \dot{\hat{\mathbf{N}}}_j$ and so the expression

$$\dot{\hat{\mathbf{N}}}_i = \sum_{m=1}^{3} W_{im} \hat{\mathbf{N}}_m \tag{2.6.7}$$

is valid provided W_{ij} are the components of a skew-symmetric tensor, $W_{ij} = -W_{ji}$. This leads to an alternative expression for the Green-Lagrange tensor:

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \sum_{\substack{m,n=1 \\ m \neq n}}^{3} \tfrac{1}{2} W_{mn} \left(\lambda_m^2 - \lambda_n^2\right) \hat{\mathbf{N}}_m \otimes \hat{\mathbf{N}}_n \tag{2.6.8}$$

Similarly, from 2.2.37, the left Cauchy-Green tensor can be expressed in terms of the principal spatial base vectors:

$$\mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \quad \dot{\mathbf{b}} = \sum_{i=1}^{3} \left\{ 2\lambda_i \dot{\lambda}_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.9}$$

Then, from inspection of 2.5.18c, $\dot{\mathbf{b}} = \mathbf{lb} + \mathbf{bl}^{\mathrm{T}}$, the velocity gradient can be expressed as {▲Problem 2}

$$\mathbf{l} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i \right\} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.7}$$

2.6.3 Rates of Change and the Relative Deformation

Just as the material time derivative of the deformation gradient is defined as

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t} \mathbf{F}(\mathbf{X}, t) = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right)$$

one can define the material time derivative of the relative deformation gradient, *cf.* §2.3.2, the rate of change *relative to the current configuration*:

$$\dot{\mathbf{F}}_t(\mathbf{x}, t) = \frac{\partial}{\partial \tau} \mathbf{F}_t(\mathbf{x}, \tau) \Big|_{\tau = t} \tag{2.6.8}$$

From 2.3.8, $\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{F}(\mathbf{X},\tau)\mathbf{F}(\mathbf{X},t)^{-1}$, so taking the derivative with respect to τ (t is now fixed) and setting $\tau = t$ gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \dot{\mathbf{F}}(\mathbf{X},t)\mathbf{F}(\mathbf{X},t)^{-1}$$

Then, from 2.5.4,

$$\mathbf{l} = \dot{\mathbf{F}}_t(\mathbf{x},t) \qquad (2.6.9)$$

as expected – the velocity gradient is the rate of change of deformation relative to the current configuration. Further, using the polar decomposition,

$$\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{R}_t(\mathbf{x},\tau)\mathbf{U}_t(\mathbf{x},\tau)$$

Differentiating with respect to τ and setting $\tau = t$ then gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \mathbf{R}_t(\mathbf{x},t)\dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t)\mathbf{U}_t(\mathbf{x},t)$$

Relative to the current configuration, $\mathbf{R}_t(\mathbf{x},t) = \mathbf{U}_t(\mathbf{x},t) = \mathbf{I}$, so, from 2.4.34,

$$\mathbf{l} = \dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t) \qquad (2.6.10)$$

With **U** symmetric and **R** skew-symmetric, $\dot{\mathbf{U}}_t(\mathbf{x},t)$, $\dot{\mathbf{R}}_t(\mathbf{x},t)$ are, respectively, symmetric and skew-symmetric, and it follows that

$$\begin{aligned} \mathbf{d} &= \dot{\mathbf{U}}_t(\mathbf{x},t) \\ \mathbf{w} &= \dot{\mathbf{R}}_t(\mathbf{x},t) \end{aligned} \qquad (2.6.11)$$

again, as expected – the rate of deformation is the instantaneous rate of stretching and the spin is the instantaneous rate of rotation.

The Corotational Derivative

The **corotational derivative** of a vector **a** is $\overset{\circ}{\mathbf{a}} \equiv \dot{\mathbf{a}} - \mathbf{w}\mathbf{a}$. Formally, it is defined through

$$\begin{aligned} \overset{\circ}{\mathbf{a}} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{R}_t(t+\Delta t)\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{R}_t(t) + \Delta t\dot{\mathbf{R}}_t(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{I} + \Delta t\mathbf{w}(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{a}(t)\right\} - \mathbf{w}(t)\mathbf{a}(t) \\ &= \dot{\mathbf{a}} - \mathbf{w}\mathbf{a} \end{aligned} \qquad (2.6.12)$$

The definition shows that the corotational derivative involves taking a vector **a** in the current configuration and rotating it with the rigid body rotation part of the motion, Fig. 2.6.1. It is this new, rotated, vector which is compared with the vector $\mathbf{a}(t+\Delta t)$, which has undergone rotation and stretch.

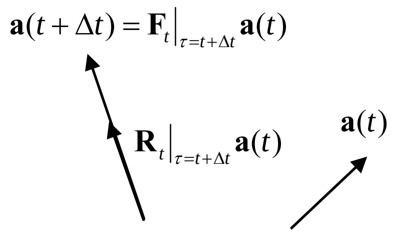

Figure 2.6.1: rotation and stretch of a vector

2.6.4 Rivlin-Ericksen Tensors

The *n*-th **Rivlin-Ericksen tensor** is defined as

$$\mathbf{A}_n(t) = \left.\frac{d^n}{d\tau^n}\mathbf{C}_t(\tau)\right|_{\tau=t}, \qquad n = 0, 1, 2, \cdots \tag{2.6.13}$$

where $\mathbf{C}_t(\tau)$ is the relative right Cauchy-Green strain. Since $\mathbf{C}_t(\tau)\big|_{\tau=t} = \mathbf{I}$, $\mathbf{A}_0 = \mathbf{I}$. To evaluate the next Rivlin-Ericksen tensor, one needs the derivatives of the relative deformation gradient; from 2.5.4, 2.3.8,

$$\frac{d}{d\tau}\mathbf{F}_t(\tau) = \frac{d}{d\tau}\left[\mathbf{F}(\tau)\mathbf{F}(t)^{-1}\right] = \mathbf{l}(\tau)\mathbf{F}(\tau)\mathbf{F}(t)^{-1} = \mathbf{l}(\tau)\mathbf{F}_t(\tau) \tag{2.6.14}$$

Then, with 2.5.5a, $d\left(\mathbf{F}_t(\tau)^{\mathrm{T}}\right)/d\tau = \mathbf{F}_t(\tau)^{\mathrm{T}}\mathbf{l}(\tau)^{\mathrm{T}}$, and

$$\begin{aligned}\mathbf{A}_1(t) &= \left[\mathbf{F}_t(\tau)^{\mathrm{T}}\left(\mathbf{l}(\tau)+\mathbf{l}(\tau)^{\mathrm{T}}\right)\mathbf{F}_t(\tau)\right]_{\tau=t} \\ &= \left(\mathbf{l}(t)+\mathbf{l}(t)^{\mathrm{T}}\right) \\ &= 2\mathbf{d}\end{aligned}$$

Thus the tensor $\mathbf{A}_1$ gives a measure of the rate of stretching of material line elements (see Eqn. 2.5.10). Similarly, higher Rivlin-Ericksen tensors give a measure of higher order stretch rates, $\ddot{\lambda}, \dddot{\lambda}$, and so on.

2.6.5 The Directional Derivative and the Material Time Derivative

The directional derivative of a function $\mathbf{T}(t)$ in the direction of an increment in *t* is, by definition (see, for example, Eqn. 1.15.27),

$$\partial_t \mathbf{T}[\Delta t] = \mathbf{T}(t + \Delta t) - \mathbf{T}(t) \tag{2.6.15}$$

or

$$\partial_t \mathbf{T}[\Delta t] = \frac{d\mathbf{T}}{dt}\Delta t \tag{2.6.16}$$

Setting $\Delta t = 1$, and using the chain rule 1.15.28,

$$\begin{aligned} \dot{\mathbf{T}} &= \partial_t \mathbf{T}[1] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\partial_t \mathbf{x}[1]\right] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\mathbf{v}\right] \end{aligned} \tag{2.6.17}$$

The material time derivative is thus equivalent to the directional derivative in the direction of the velocity vector.

2.6.6 Problems

1. Derive the relations 2.6.2.
2. Use 2.6.9 to verify 2.5.18, $\dot{\mathbf{b}} = \mathbf{lb} + \mathbf{bl}^{\mathrm{T}}$.