# 2.5 Deformation Rates

In this section, rates of change of the deformation tensors introduced earlier, **F**, **C**, **E**, etc., are evaluated, and special tensors used to measure deformation rates are discussed, for example the velocity gradient **l**, the rate of deformation **d** and the spin tensor **w**.

## 2.5.1 The Velocity Gradient

The **velocity gradient** is used as a measure of the rate at which a material is deforming.

Consider two fixed neighbouring points, $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, Fig. 2.5.1. The velocities of the material particles at these points at any given time instant are $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x} + d\mathbf{x})$, and

$$\mathbf{v}(\mathbf{x} + d\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x},$$

The relative velocity between the points is

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x} \equiv \mathbf{l} d\mathbf{x} \tag{2.5.1}$$

with **l** defined to be the (spatial) velocity gradient,

$$\boxed{\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \operatorname{grad} \mathbf{v}, \qquad l_{ij} = \frac{\partial v_i}{\partial x_j}} \qquad \textbf{Spatial Velocity Gradient} \tag{2.5.2}$$

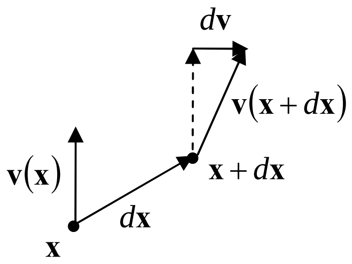


**Figure 2.5.1: velocity gradient**

Expression 2.5.1 emphasises the tensorial character of the spatial velocity gradient, mapping as it does one vector into another. Its physical meaning will become clear when it is decomposed into its symmetric and skew-symmetric parts below.

The spatial velocity gradient is commonly used in both solid and fluid mechanics. Less commonly used is the material velocity gradient, which is related to the rate of change of the deformation gradient:

$$\operatorname{Grad} \mathbf{V} = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial \mathbf{X}}\right) = \dot{\mathbf{F}} \tag{2.5.3}$$

and use has been made of the fact that, since **X** and *t* are independent variables, material time derivatives and material gradients commute.

## 2.5.2 Material Derivatives of the Deformation Gradient

The spatial velocity gradient may be written as

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}}{\partial t}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}}$$

or $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ so that the material derivative of **F** can be expressed as

$$\boxed{\dot{\mathbf{F}} = \mathbf{l}\,\mathbf{F}} \qquad \textbf{Material Time Derivative of the Deformation Gradient} \qquad (2.5.4)$$

Also, it can be shown that {▲Problem 1}

$$\boxed{\begin{aligned} \overline{\mathbf{F}^{\mathrm{T}}}^{\boldsymbol{\cdot}} &= \dot{\mathbf{F}}^{\mathrm{T}} \\ \overline{\mathbf{F}^{-1}}^{\boldsymbol{\cdot}} &= -\mathbf{F}^{-1}\mathbf{l} \\ \overline{\mathbf{F}^{-\mathrm{T}}}^{\boldsymbol{\cdot}} &= -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} \end{aligned}} \qquad (2.5.5)$$

## 2.5.3 The Rate of Deformation and Spin Tensors

The velocity gradient can be decomposed into a symmetric tensor and a skew-symmetric tensor as follows (see §1.10.10):

$$\boxed{\mathbf{l} = \mathbf{d} + \mathbf{w}} \qquad (2.5.6)$$

where **d** is the **rate of deformation tensor** (or **rate of stretching tensor**) and **w** is the **spin tensor** (or **rate of rotation**, or **vorticity tensor**), defined by

$$\boxed{\begin{aligned} \mathbf{d} &= \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad & d_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ \mathbf{w} &= \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad & w_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \end{aligned}} \qquad \textbf{Rate of Deformation and Spin Tensors}$$

(2.5.7)

The physical meaning of these tensors is next examined.

### The Rate of Deformation

Consider first the rate of deformation tensor **d** and note that

$$\mathbf{l}d\mathbf{x} = d\mathbf{v} = \frac{d}{dt}(d\mathbf{x}) \tag{2.5.8}$$

The rate at which the square of the length of $d\mathbf{x}$ is changing is then

$$\begin{aligned} &\frac{d}{dt}\left(|d\mathbf{x}|^2\right) = 2|d\mathbf{x}|\frac{d}{dt}\left(|d\mathbf{x}|\right), \\ &\frac{d}{dt}\left(|d\mathbf{x}|^2\right) = \frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = 2d\mathbf{x}\cdot\frac{d}{dt}(d\mathbf{x}) = 2d\mathbf{x}\,\mathbf{l}\,d\mathbf{x} = 2d\mathbf{x}\,\mathbf{d}\,d\mathbf{x} \end{aligned} \tag{2.5.9}$$

the last equality following from 2.5.6 and 1.10.31e. Dividing across by $2|d\mathbf{x}|^2$, then leads to

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \hat{\mathbf{n}}\,\mathbf{d}\,\hat{\mathbf{n}}} \qquad \textbf{Rate of stretching per unit stretch in the direction } \hat{\mathbf{n}} \tag{2.5.10}$$

where $\lambda = |d\mathbf{x}|/|d\mathbf{X}|$ is the stretch and $\hat{\mathbf{n}} = d\mathbf{x}/|d\mathbf{x}|$ is a unit normal in the direction of $d\mathbf{x}$. Thus the rate of deformation **d** gives the rate of stretching of line elements. The diagonal components of **d**, for example

$$d_{11} = \mathbf{e}_1\mathbf{d}\mathbf{e}_1,$$

represent unit rates of extension in the coordinate directions.

Note that these are *instantaneous* rates of extension, in other words, they are rates of extensions of elements in the current configuration at the current time; they are not a measure of the rate at which a line element in the original configuration changed into the corresponding line element in the current configuration.

Note:

- Eqn. 2.5.10 can also be derived as follows: let $\hat{\mathbf{N}}$ be a unit normal in the direction of $d\mathbf{X}$, and $\hat{\mathbf{n}}$ be the corresponding unit normal in the direction of $d\mathbf{x}$. Then $\hat{\mathbf{n}}|d\mathbf{x}| = \mathbf{F}\hat{\mathbf{N}}|d\mathbf{X}|$, or $\hat{\mathbf{n}}\lambda = \mathbf{F}\hat{\mathbf{N}}$. Differentiating gives $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \dot{\mathbf{F}}\hat{\mathbf{N}} = \mathbf{l}\mathbf{F}\hat{\mathbf{N}}$ or $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \mathbf{l}\hat{\mathbf{n}}\lambda$. Contracting both sides with $\hat{\mathbf{n}}$ leads to $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} + \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}(\dot{\lambda}/\lambda) = \hat{\mathbf{n}}\mathbf{l}\hat{\mathbf{n}}$. But $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1 \rightarrow d(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})dt = 0$ so, by the chain rule, $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} = 0$ (confirming that a vector $\hat{\mathbf{n}}$ of constant length is orthogonal to a change in that vector $d\hat{\mathbf{n}}$), and the result follows

Consider now the rate of change of the angle $\theta$ between two vectors $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. Using 2.5.8 and 1.10.3d,

$$\begin{aligned}\frac{d}{dt}\left(d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}\right)&=\frac{d}{dt}\left(d\mathbf{x}^{(1)}\right)\cdot d\mathbf{x}^{(2)}+d\mathbf{x}^{(1)}\cdot\frac{d}{dt}\left(d\mathbf{x}^{(2)}\right)\\&=\mathbf{l}d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}+d\mathbf{x}^{(1)}\cdot\mathbf{l}d\mathbf{x}^{(2)}\\&=\left(\mathbf{l}+\mathbf{l}^{\mathrm{T}}\right)d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}\\&=2\,d\mathbf{x}^{(1)}\mathbf{d}d\mathbf{x}^{(2)}\end{aligned} \tag{2.5.11}$$

which reduces to 2.5.9 when $d\mathbf{x}^{(1)}=d\mathbf{x}^{(2)}$. An alternative expression for this dot product is

$$\begin{aligned}\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\cos\theta\right)&=\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)\left|d\mathbf{x}^{(2)}\right|\cos\theta+\frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)\left|d\mathbf{x}^{(1)}\right|\cos\theta-\sin\theta\,\dot{\theta}\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\\&=\left(\frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)}{\left|d\mathbf{x}^{(1)}\right|}\cos\theta+\frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)}{\left|d\mathbf{x}^{(2)}\right|}\cos\theta-\sin\theta\,\dot{\theta}\right)\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\end{aligned} \tag{2.5.12}$$

Equating 2.5.11 and 2.5.12 leads to

$$2\,\hat{\mathbf{n}}_1\,\mathbf{d}\,\hat{\mathbf{n}}_2=\left(\frac{\dot{\lambda}_1}{\lambda_1}+\frac{\dot{\lambda}_2}{\lambda_2}\right)\cos\theta-\sin\theta\,\dot{\theta} \tag{2.5.13}$$

where $\lambda_i=\left|d\mathbf{x}^{(i)}\right|/\left|d\mathbf{X}^{(i)}\right|$ is the stretch and $\hat{\mathbf{n}}_i=d\mathbf{x}^{(i)}/\left|d\mathbf{x}^{(i)}\right|$ is a unit normal in the direction of $d\mathbf{x}^{(i)}$.

It follows from 2.5.13 that the off-diagonal terms of the rate of deformation tensor represent **shear rates**: the rate of change of the right angle between line elements aligned with the coordinate directions. For example, taking the base vectors $\mathbf{e}_1=\hat{\mathbf{n}}_1$, $\mathbf{e}_2=\hat{\mathbf{n}}_2$, 2.5.13 reduces to

$$d_{12}=-\frac{1}{2}\dot{\theta}_{12} \tag{2.5.14}$$

where $\theta_{12}$ is the instantaneous right angle between the axes in the current configuration.

### The Spin

Consider now the spin tensor $\mathbf{w}$; since it is skew-symmetric, it can be written in terms of its axial vector $\boldsymbol{\omega}$ (Eqn. 1.10.34), called the **angular velocity vector**:

$$\begin{aligned}\boldsymbol{\omega} &= -w_{23}\mathbf{e}_1 + w_{13}\mathbf{e}_2 - w_{12}\mathbf{e}_3 \\ &= \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \\ &= \frac{1}{2}\operatorname{curl}\mathbf{v}\end{aligned} \tag{2.5.15}$$

(The vector $2\boldsymbol{\omega}$ is called the **vorticity** (or **spin**) **vector**.) Thus when **d** is zero, the motion consists of a rotation about some axis at angular velocity $\omega = |\boldsymbol{\omega}|$ (*cf.* the end of §1.10.11), with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, **r** measured from a point on the axis, and $\mathbf{wr} = \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}$.

On the other hand, when $\mathbf{l} = \mathbf{d}$, $\mathbf{w} = \mathbf{0}$, one has $\boldsymbol{\omega} = \mathbf{o}$, and the motion is called **irrotational**.

### Example (Shear Flow)

Consider a **simple shear flow** in which the velocity profile is "triangular" as shown in Fig. 2.5.2. This type of flow can be generated (at least approximately) in many fluids by confining the fluid between plates a distance *h* apart, and by sliding the upper plate over the lower one at constant velocity *V*. If the material particles adjacent to the upper plate have velocity $V\mathbf{e}_1$, then the velocity field is $\mathbf{v} = \dot{\gamma} x_2 \mathbf{e}_1$, where $\dot{\gamma} = V/h$. This is a steady flow ($\partial \mathbf{v}/\partial t = \mathbf{0}$); at any given point, there is no change over time. The velocity gradient is $\mathbf{l} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2$ and the acceleration of material particles is zero: $\mathbf{a} = \mathbf{lv} = \mathbf{0}$. The rate of deformation and spin are

$$\mathbf{d} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{w} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and, from 2.5.14, $\dot{\gamma} = -\dot{\theta}_{12}$, the rate of change of the angle shown in Fig. 2.5.2.

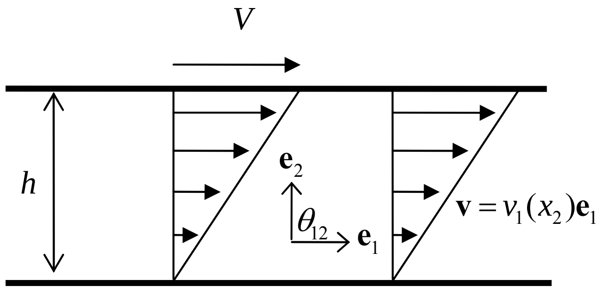


**Figure 2.5.2: shear flow**

The eigenvalues of **d** are $\lambda = 0, \pm\dot{\gamma}/2$ ($\det \mathbf{d} = 0$) and the principal invariants, Eqn. 1.11.17, are $\mathrm{I}_\mathbf{d} = 0$, $\mathrm{II}_\mathbf{d} = -\frac{1}{4}\dot{\gamma}^2$, $\mathrm{III}_\mathbf{d} = 0$. For $\lambda = +\dot{\gamma}/2$, the eigenvector is $\mathbf{n}_1 = [1 \quad 1 \quad 0]^T$ and for $\lambda = -\dot{\gamma}/2$, it is $\mathbf{n}_2 = [-1 \quad 1 \quad 0]^T$ (for $\lambda = 0$ it is $\mathbf{e}_3$). (The eigenvalues and eigenvectors of **w** are complex.) Relative to the basis of eigenvectors,

$$\mathbf{d} = \begin{bmatrix} \dot{\gamma}/2 & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so at $45^\circ$ there is an instantaneous pure rate of stretching/contraction of material.

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## 2.5.4 Other Rates of Strain Tensors

From 2.2.9, 2.2.22,

$$\frac{1}{2}\frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = d\mathbf{X}\frac{1}{2}\dot{\mathbf{C}}d\mathbf{X} = d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} \tag{2.5.16}$$

This can also be written in terms of spatial line elements:

$$d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} = d\mathbf{x}\left[\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1}\right]d\mathbf{x} \tag{2.5.17}$$

But from 2.5.9, these also equal $d\mathbf{x}\mathbf{d}d\mathbf{x}$, which leads to expressions for the material time derivatives of the right Cauchy-Green and Green-Lagrange strain tensors (also given here are expressions for the time derivatives of the left Cauchy-Green and Euler-Almansi tensors {▲Problem 3})

$$\boxed{\begin{aligned} &\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ &\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ &\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}} \\ &\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l} \end{aligned}} \tag{2.5.18}$$

Note that

$$\int \dot{\mathbf{E}}\, dt = \int d\mathbf{E}$$

so that the integral of the rate of Green-Lagrange strain is path independent and, in particular, the integral of $\dot{\mathbf{E}}$ around any closed loop (so that the final configuration is the same as the initial configuration) is zero. However, in general, the integral of the rate of deformation,

$$\int \mathbf{d}dt$$

is not independent of the path – there is no universal function $\mathbf{h}$ such that $\mathbf{d} = d\mathbf{h}/dt$ with $\int \mathbf{d}dt = \int d\mathbf{h}$. Thus the integral $\int \mathbf{d}dt$ over a closed path may be non-zero, and hence the integral of the rate of deformation is not a good measure of the total strain.

### The Hencky Strain

The Hencky strain is, Eqn. 2.2.37, $\mathbf{h} = \sum_{i=1}^{3} (\ln \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$ , where $\mathbf{n}_i$ are the principal spatial axes. Thus, if the principal spatial axes do not change with time, $\dot{\mathbf{h}} = \sum_{i=1}^{3} (\dot{\lambda}_i / \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$ . With the left stretch $\mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$ , it follows that (and similarly for the corresponding material tensors), $\dot{\mathbf{H}} \equiv \dot{\overline{\ln \mathbf{U}}} = \dot{\mathbf{U}}\mathbf{U}^{-1}$, $\dot{\mathbf{h}} \equiv \dot{\overline{\ln \mathbf{v}}} = \dot{\mathbf{v}}\mathbf{v}^{-1}$ .

For example, consider an extension in the coordinate directions, so $\mathbf{F} = \mathbf{U} = \mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i$ . The motion and velocity are

$$x_i = \lambda_i X_i, \quad \dot{x}_i = \dot{\lambda}_i X_i = \frac{\dot{\lambda}_i}{\lambda_i} x_i \qquad (\text{no sum})$$

so $d_i = \dot{\lambda}_i / \lambda_i$ (no sum), and $\mathbf{d} = \dot{\mathbf{h}}$ . Further, $\mathbf{h} = \int \mathbf{d} dt$ . Note that, as mentioned above, this expression does not hold in general, but does in this case of uniform extension.

## 2.5.5 Material Derivatives of Line, Area and Volume Elements

The material derivative of a line element $d(d\mathbf{x})/dt$ has been derived (defined) through 2.4.8. For area and volume elements, it is necessary first to evaluate the material derivative of the Jacobian determinant *J*. From the chain rule, one has (see Eqns 1.15.11, 1.15.7)

$$\dot{J} = \frac{d}{dt}\left(J(\mathbf{F})\right) = \frac{\partial J}{\partial \mathbf{F}} : \dot{\mathbf{F}} = J\mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} \qquad (2.5.19)$$

Hence {▲Problem 4}

$$\boxed{\begin{aligned} \dot{J} &= J\,\mathrm{tr}(\mathbf{l}) \\ &= J\,\mathrm{tr}(\mathrm{grad}\,\mathbf{v}) \\ &= J\mathrm{div}\mathbf{v} \end{aligned}} \qquad (2.5.20)$$

Since $\mathbf{l} = \mathbf{d} + \mathbf{w}$ and $\mathrm{tr}\mathbf{w} = 0$, it also follows that $\dot{J} = J\mathrm{tr}\mathbf{d}$.

As mentioned earlier, an isochoric motion is one for which the volume is constant – thus any of the following statements characterise the necessary and sufficient conditions for an isochoric motion:

$$J = 1, \quad \dot{J} = 0, \quad \mathrm{div}\mathbf{v} = 0, \quad \mathrm{tr}\,\mathbf{d} = 0, \quad \mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} = 0 \qquad (2.5.21)$$

Applying Nanson's formula 2.2.59, the material derivative of an area vector element is {▲Problem 6}

$$\boxed{\frac{d}{dt}(\hat{\mathbf{n}}ds) = (\mathrm{div}\mathbf{v} - \mathbf{l}^{\mathrm{T}})\hat{\mathbf{n}}ds} \tag{2.5.22}$$

Finally, from 2.2.53, the material time derivative of a volume element is

$$\boxed{\frac{d}{dt}(dv) = \frac{d}{dt}(JdV) = \dot{J}dV = \mathrm{div}\mathbf{v}\,dv} \tag{2.5.23}$$

### Example (Shear and Stretch)

Consider a sample of material undergoing the following motion, Fig. 2.4.3.

$$\begin{aligned} x_1 &= X_1 + k\lambda X_2 \\ x_2 &= \lambda X_2 \\ x_3 &= X_3 \end{aligned}, \qquad \begin{aligned} X_1 &= x_1 - kx_2 \\ X_2 &= \frac{1}{\lambda}x_2 \\ X_3 &= x_3 \end{aligned}$$

with $\lambda = \lambda(t)$, $k = k(t)$.

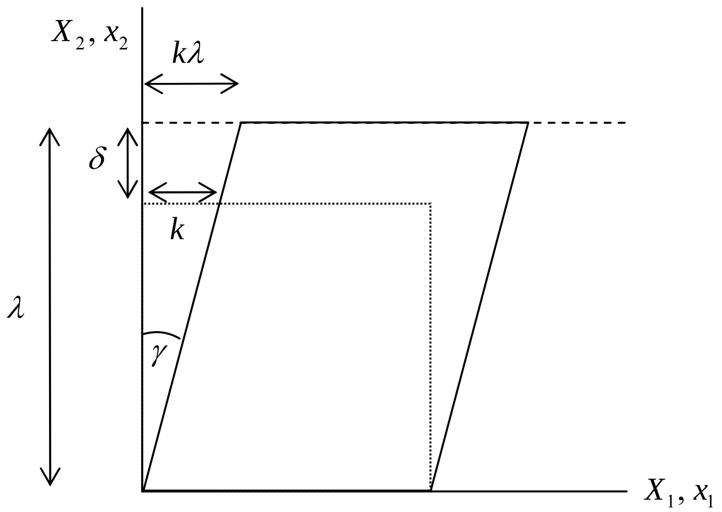


**Figure 2.4.3: shear and stretch**

The deformation gradient and material strain tensors are

$$\mathbf{F} = \begin{bmatrix} 1 & k\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & k\lambda & 0 \\ k\lambda & (1+k^2)\lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & \frac{1}{2}k\lambda & 0 \\ \frac{1}{2}k\lambda & \frac{1}{2}(\lambda^2(1+k^2)-1) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the Jacobian $J = \det \mathbf{F} = \lambda$, and the spatial strain tensors are

$$\mathbf{b} = \begin{bmatrix} 1+k^2\lambda^2 & k\lambda^2 & 0 \\ k\lambda^2 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} 0 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & \frac{1}{2}\frac{(1-k^2)\lambda^2 - 1}{\lambda^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This deformation can also be expressed as a stretch followed by a simple shear:

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The velocity is

$$\mathbf{V} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} (\dot{k}\lambda + k\dot{\lambda})X_2 \\ \dot{\lambda}X_2 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} (\dot{k} + k(\dot{\lambda}/\lambda))x_2 \\ (\dot{\lambda}/\lambda)x_2 \\ 0 \end{bmatrix}$$

The velocity gradient is

$$\mathbf{l} = \frac{d\mathbf{v}}{d\mathbf{x}} = \begin{bmatrix} 0 & \dot{k} + k(\dot{\lambda}/\lambda) & 0 \\ 0 & \dot{\lambda}/\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the rate of deformation and spin are

$$\mathbf{d} = \begin{bmatrix} 0 & \frac{1}{2}\left[\dot{k} + k(\dot{\lambda}/\lambda)\right] & 0 \\ \frac{1}{2}\left[\dot{k} + k(\dot{\lambda}/\lambda)\right] & \dot{\lambda}/\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 & \frac{1}{2}\left[\dot{k} + k(\dot{\lambda}/\lambda)\right] & 0 \\ -\frac{1}{2}\left[\dot{k} + k(\dot{\lambda}/\lambda)\right] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also

$$\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{d}\mathbf{F} = \begin{bmatrix} 0 & \lambda\dot{k} + k\dot{\lambda} & 0 \\ \lambda\dot{k} + k\dot{\lambda} & 2\lambda\left(k\lambda\dot{k} + (k^2+1)\dot{\lambda}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As expected, from 2.5.20,

$$\dot{J} = J\mathrm{tr}(\mathbf{d}) = J(\dot{\lambda}/\lambda) = \dot{\lambda}$$

■

## 2.5.6 Problems

1. (a) Differentiate the relation $\mathbf{I} = \mathbf{F}\mathbf{F}^{-1}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, to derive 2.5.5b,

   $\dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\mathbf{l}$.

   (b) Differentiate the relation $\mathbf{I} = \mathbf{F}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, and 1.10.3e to derive 2.5.5c, $\dot{\overline{\mathbf{F}^{-\mathrm{T}}}} = -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$.

2. For the velocity field

   $$v_1 = x_1^2 x_2, \quad v_2 = 2x_2^2 x_3, \quad v_3 = 3x_1x_2x_3$$

   determine the rate of stretching per unit stretch at (2,0,1) in the direction of the unit vector

   $$(4\mathbf{e}_1 - 3\mathbf{e}_2)/5$$

   And in the direction of $\mathbf{e}_1$?

3. (a) Derive the relation 2.5.18a, $\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$ directly from $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$

   (b) Use the definitions $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{e} = (\mathbf{I} - \mathbf{b}^{-1})/2$ to derive the relations 2.5.18c,d: $\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}}$, $\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l}$

4. Use 2.5.4, 2.5.19, 1.10.3h, 1.10.6, to derive 2.5.20.

5. For the motion $x_1 = 3X_1t - t^2,\ x_2 = X_1 + X_2t,\ x_3 = tX_3$, verify that $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$. What is the ratio of the volume element currently occupying $(1,1,1)$ to its volume in the undeformed configuration? And what is the rate of change of this volume element, per unit current volume?

6. Use Nanson's formula 2.2.59, the product rule of differentiation, and 2.5.20, 2.5.5c, to derive the material time derivative of a vector area element, 2.5.22 (note that $\hat{\mathbf{N}}$, a unit normal in the undeformed configuration, is constant).