2.3 Deformation and Strain: Further Topics

2.3.1 Volumetric and Isochoric Deformations

When analysing materials which are only slightly incompressible, it is useful to decompose the deformation gradient multiplicatively, according to

$$\mathbf{F} = \left(J^{1/3}\mathbf{I}\right)\overline{\mathbf{F}} = J^{1/3}\overline{\mathbf{F}} \tag{2.3.1}$$

From this definition {▲Problem 1},

$$\det \overline{\mathbf{F}} = 1 \tag{2.3.2}$$

and so $\overline{\mathbf{F}}$ characterises a volume preserving (**distortional** or **isochoric**) deformation. The tensor $J^{1/3}\mathbf{I}$ characterises the volume-changing (**dilational** or **volumetric**) component of the deformation, with $\det\left(J^{1/3}\mathbf{I}\right) = \det\mathbf{F} = J$.

This concept can be carried on to other kinematic tensors. For example, with $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$,

$$\mathbf{C} = J^{2/3}\overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}} \equiv J^{2/3}\overline{\mathbf{C}}. \tag{2.3.3}$$

$\overline{\mathbf{F}}$ and $\overline{\mathbf{C}}$ are called the **modified deformation gradient** and the **modified right Cauchy-Green tensor**, respectively. The square of the stretch is given by

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}} = J^{2/3}\left\{d\hat{\mathbf{X}}\overline{\mathbf{C}}d\hat{\mathbf{X}}\right\} \tag{2.3.4}$$

so that $\lambda = J^{1/3}\overline{\lambda}$, where $\overline{\lambda}$ is the **modified stretch**, due to the action of $\overline{\mathbf{C}}$. Similarly, the **modified principal stretches** are

$$\overline{\lambda}_i = J^{-1/3}\lambda_i, \quad i = 1,2,3 \tag{2.3.5}$$

with

$$\det\overline{\mathbf{F}} = \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3 = 1 \tag{2.3.6}$$

The case of simple shear discussed earlier is an example of an isochoric deformation, in which the deformation gradient and the modified deformation gradient coincide, $J^{1/3}\mathbf{I} = \mathbf{I}$.

2.3.2 Relative Deformation

It is usual to use the configuration at $(\mathbf{X}, t = 0)$ as the reference configuration, and define quantities such as the deformation gradient relative to this reference configuration. As mentioned, any configuration can be taken to be the reference configuration, and a new deformation gradient can be constructed with respect to this new reference configuration. Further, the reference configuration does not have to be fixed, but could be moving also.

In many cases, it is useful to choose the *current* configuration $(\mathbf{x}, t)$ to be the reference configuration, for example when evaluating rates of change of kinematic quantities (see later). To this end, introduce a third configuration: this is the configuration at some time $t = \tau$ and the position of a material particle $\mathbf{X}$ here is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}(\mathbf{X}, \tau)$, where $\boldsymbol{\chi}$ is the motion function. The deformation at this time τ relative to the *current* configuration is called the **relative deformation**, and is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}_{(t)}(\mathbf{x}, \tau)$, as illustrated in Fig. 2.3.1.

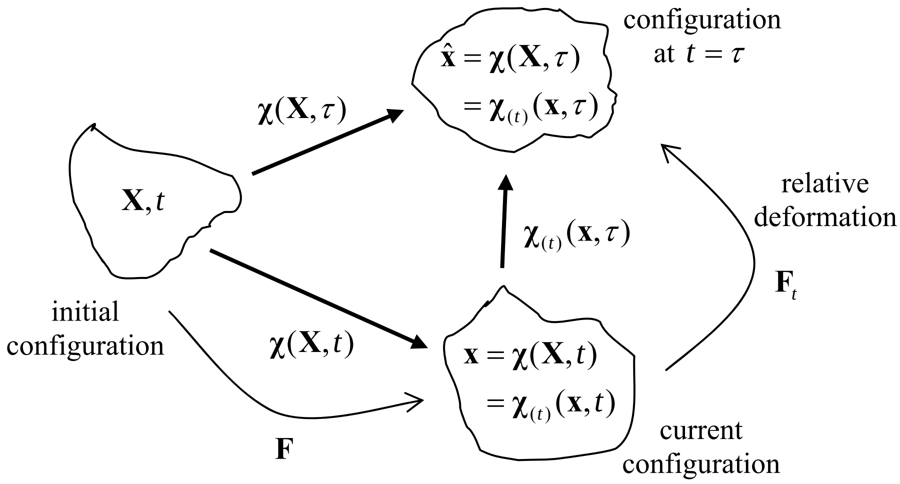

Figure 2.3.1: the relative deformation

The **relative deformation gradient** $\mathbf{F}_t$ is defined through

$$d\hat{\mathbf{x}} = \mathbf{F}_t(\mathbf{x}, \tau)\, d\mathbf{x}, \qquad \mathbf{F}_t = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \tag{2.3.7}$$

Also, since $d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$ and $d\hat{\mathbf{x}} = \mathbf{F}(\mathbf{X}, \tau)\, d\mathbf{X}$, one has the relation

$$\mathbf{F}(\mathbf{X}, \tau) = \mathbf{F}_t(\mathbf{x}, \tau)\, \mathbf{F}(\mathbf{X}, t) \tag{2.3.8}$$

Similarly, relative strain measures can be defined, for example the relative right Cauchy-Green strain tensor is

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^{\mathrm{T}}\, \mathbf{F}_t(\tau) \tag{2.3.9}$$

Example

Consider the two-dimensional motion

$$x_1 = X_1 e^t, \quad x_2 = X_2(t+1)$$

Inverting these gives the spatial description $X_1 = x_1 e^{-t}$, $X_2 = x_2/(t+1)$, and the relative deformation is

$$\begin{aligned} \hat{x}_1(\mathbf{x},\tau) &= X_1 e^{\tau} = x_1 e^{\tau - t} \\ \hat{x}_2(\mathbf{x},\tau) &= X_2(\tau+1) = x_2(\tau+1)/(t+1) \end{aligned}$$

The deformation gradients are

$$\begin{aligned} \mathbf{F}(\mathbf{X},t) &= \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j = e^t \mathbf{e}_1 \otimes \mathbf{E}_1 + (t+1)\mathbf{e}_2 \otimes \mathbf{E}_2 \\ \mathbf{F}_t(\mathbf{x},\tau) &= \frac{\partial \hat{x}_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = e^{\tau - t}\mathbf{e}_1 \otimes \mathbf{e}_1 + (\tau+1)/(t+1)\mathbf{e}_2 \otimes \mathbf{e}_2 \end{aligned}$$

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2.3.3 Derivatives of the Stretch

In this section, some useful formulae involving the derivatives of the stretches with respect to the Cauchy-Green strain tensors are derived.

Derivatives with respect to b

First, take the stretches to be functions of the left Cauchy-Green strain **b**. Write **b** using the spatial principal directions $\hat{\mathbf{n}}_i$ as a basis, 2.2.37, so that the total differential can be expressed as

$$d\mathbf{b} = \sum_{i=1}^{3} 2\lambda_i d\lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \left[d\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \hat{\mathbf{n}}_i \otimes d\hat{\mathbf{n}}_i \right] \tag{2.3.10}$$

Since $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = \delta_{ij}$, then

$$\hat{\mathbf{n}}_i d\mathbf{b} \hat{\mathbf{n}}_i = 2\lambda_i d\lambda_i + \lambda_i^2 \left[\hat{\mathbf{n}}_i \cdot d\hat{\mathbf{n}}_i + d\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_i \right] = 2\lambda_i d\lambda_i \quad \text{(no sum over } i\text{)} \tag{2.3.11}$$

This last follows since the change in a vector of constant length is always orthogonal to the vector itself (as in the curvature analysis of §1.6.2). Using the property $\mathbf{uTv} = \mathbf{T} : (\mathbf{u} \otimes \mathbf{v})$, one has (summing over the *k* but not over the *i*; here $d\lambda_k / d\lambda_i = \delta_{ik}$)

$$d\mathbf{b} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \equiv \frac{\partial \mathbf{b}}{\partial \lambda_k} d\lambda_k : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 2\lambda_i d\lambda_i \quad \to \quad \frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 1 \tag{2.3.12}$$

Then, since $\partial \mathbf{b} / \partial \lambda_i : \partial \lambda_i / \partial \mathbf{b}$ is also equal to 1, one has

$$\frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i}:(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = \frac{\partial \mathbf{b}}{\partial \lambda_i}:\frac{\partial \lambda_i}{\partial \mathbf{b}} \quad \rightarrow \quad \frac{\partial \lambda_i}{\partial \mathbf{b}} = \frac{1}{2\lambda_i}(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \tag{2.3.13}$$

The chain rule then gives the second derivative.

The above analysis is for distinct principal stretches. When $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$, then $\mathbf{b} = \lambda^2 \mathbf{I}$, $d\mathbf{b} = 2\lambda d\lambda \mathbf{I}$. Also, $d\mathbf{b} = 3(\partial \mathbf{b}/\partial \lambda)d\lambda$, so $3(\partial \mathbf{b}/\partial \lambda) = 2\lambda \mathbf{I}$, or

$$3\frac{\partial \mathbf{b}}{\partial \lambda}:\frac{\partial \lambda}{\partial \mathbf{b}} = 2\lambda \mathbf{I}:\frac{\partial \lambda}{\partial \mathbf{b}} \tag{2.3.14}$$

But $\partial \mathbf{b}/\partial \lambda : \partial \lambda / \partial \mathbf{b} = 1$ and $3 = \mathbf{I}:\mathbf{I}$, and so in this case, $\partial \lambda / \partial \mathbf{b} = \mathbf{I}/2\lambda$.

A similar calculation can be carried out for two equal eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. In summary,

$$\boxed{\begin{array}{lr}
\dfrac{\partial \lambda_i}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_i}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \qquad (\text{no sum over } i) & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \\
\left.\begin{array}{l}\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}(\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2) \\ \dfrac{\partial \lambda_3}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_3}(\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3)\end{array}\right. & \lambda_1 = \lambda_2 = \lambda \neq \lambda_3 \\
\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}\sum_{i=1}^{3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \dfrac{1}{2\lambda}\mathbf{I} & \lambda_1 = \lambda_2 = \lambda_3 = \lambda \\
\dfrac{\partial^2 \lambda_i}{\partial \mathbf{b}^2} = -\dfrac{1}{4\lambda_i^3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \quad (\text{no sum over } i) & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1
\end{array}} \tag{2.3.15}$$

Derivatives with respect to C

The stretch can also be considered to be a function of the right Cauchy-Green strain **C**. The derivatives of the stretches with respect to **C** can be found in exactly the same way as for the left Cauchy-Green strain. The results are the same as given in 2.3.15 except that, referring to 2.2.37, **b** is replaced by **C** and $\hat{\mathbf{n}}$ is replaced by $\hat{\mathbf{N}}$.

2.3.4 The Directional Derivative of Kinematic Quantities

The directional derivative of vectors and tensors was introduced in §1.6.11 and §1.15.4. Taking directional derivatives of kinematic quantities is often very useful, for example in linearising equations in order to apply numerical solution algorithms

The Deformation Gradient

First, consider the deformation gradient as a function of the current position $\mathbf{x}$ (or motion $\boldsymbol{\chi}$) and examine its value at $\mathbf{x} + \mathbf{a}$:

$$\mathbf{F}(\mathbf{x}+\mathbf{a}) = \mathbf{F}(\mathbf{x}) + \partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] + o(|\mathbf{a}|) \tag{2.3.16}$$

The directional derivative $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] = (\partial\mathbf{F}/\partial\mathbf{x})\mathbf{a}$ can be expressed as

$$\begin{aligned}
\partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{x}+\varepsilon\mathbf{a}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial(\mathbf{x}+\varepsilon\mathbf{a})}{\partial\mathbf{X}} \\
&= \operatorname{Grad}\mathbf{a} \\
&= (\operatorname{grad}\mathbf{a})\mathbf{F}
\end{aligned} \tag{2.3.17}$$

the last line resulting from 2.2.8b. It follows that the directional derivative of the deformation gradient in the direction of a displacement vector $\mathbf{u}$ from the *current* configuration is

$$\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = (\operatorname{grad}\mathbf{u})\mathbf{F} \tag{2.3.18}$$

On the other hand, consider the deformation gradient as a function of $\mathbf{X}$ and examine its value at $\mathbf{X}+\mathbf{A}$:

$$\mathbf{F}(\mathbf{X}+\mathbf{A}) = \mathbf{F}(\mathbf{X}) + \partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] \tag{2.3.19}$$

and now

$$\begin{aligned}
\partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{X}+\varepsilon\mathbf{A}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}\mathbf{x}(\mathbf{X}+\varepsilon\mathbf{A}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}(\mathbf{x}+\mathbf{F}\varepsilon\mathbf{A}) \\
&= \operatorname{Grad}(\mathbf{F}\mathbf{A}) \\
&= \operatorname{Grad}\mathbf{a}
\end{aligned} \tag{2.3.20}$$

where $\mathbf{a} = \mathbf{F}\mathbf{A}$.

Other Kinematic Quantities

The directional derivative of the Green-Lagrange strain, the right and left Cauchy-Green tensors and the Jacobian in the direction of a displacement $\mathbf{u}$ from the current configuration are {▲Problem 2}

$$\begin{aligned}
\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] &= \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}] &= 2\mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}] &= (\mathrm{grad}\mathbf{u})\mathbf{b} + \mathbf{b}(\mathrm{grad}\mathbf{u})^{\mathrm{T}} \\
\partial_{\mathbf{x}}J[\mathbf{u}] &= J\mathrm{div}\mathbf{u}
\end{aligned} \tag{2.3.21}$$

where $\boldsymbol{\varepsilon}$ is the small-strain tensor, 2.2.48.

The directional derivative is also useful for deriving various relations between the kinematic variables. For example, for an arbitrary vector **a**, using the chain rule 1.15.28, 2.3.20, 1.15.24, the trace relations 1.10.10e and 1.10.10b, and 2.2.8b, 1.14.9,

$$\begin{aligned}
(\mathrm{Grad}J)\cdot\mathbf{a} &= \partial_{\mathbf{X}}J[\mathbf{a}] \\
&= \partial_{\mathbf{F}}J[\partial_{\mathbf{X}}\mathbf{F}[\mathbf{a}]] \\
&= \partial_{\mathbf{F}}J[\mathrm{Grad}(\mathbf{Fa})] \\
&= J\mathbf{F}^{-\mathrm{T}} : \mathrm{Grad}(\mathbf{Fa}) \\
&= J\mathrm{tr}(\mathbf{F}^{-1}\mathrm{Grad}(\mathbf{Fa})) \\
&= J\mathrm{tr}(\mathrm{Grad}(\mathbf{Fa})\mathbf{F}^{-1}) \\
&= J\mathrm{tr}(\mathrm{grad}(\mathbf{Fa})) \\
&= J\,\mathrm{div}(\mathbf{Fa})
\end{aligned} \tag{2.3.22}$$

so that, from 1.14.16b with **a** constant,

$$\boxed{\mathrm{Grad}J = J\mathrm{div}\mathbf{F}^{\mathrm{T}}} \tag{2.3.23}$$

2.3.5 Problems

1. Use 1.10.16c to show that $\det\overline{\mathbf{F}} = 1$.
2. (a) use the relation $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$, Eqn. 2.3.18, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = (\mathrm{grad}\mathbf{u})\mathbf{F}$, and the product rule of differentiation to derive 2.3.21a, $\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] = \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F}$, where $\boldsymbol{\varepsilon}$ is the small strain tensor.
   (b) evaluate $\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}]$ (in terms of **F** and $\boldsymbol{\varepsilon}$, the small strain tensor)
   (c) evaluate $\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}]$ (in terms of $\mathrm{grad}\mathbf{u}$ and **b**)
   (d) evaluate $\partial_{\mathbf{x}}J[\mathbf{u}]$ (in terms of J and $\mathrm{div}\mathbf{u}$; use the chain rule $\partial_{\mathbf{x}}J[\mathbf{u}] = \partial_{\mathbf{F}}\hat{J}[\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}]]$, with $\hat{J}(\mathbf{F}) = \det\mathbf{F}$, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = \mathrm{Grad}\mathbf{u}$)