8 2-D FEM: Poisson's Equation

Here, the FEM solution to the 2D Poisson equation is considered.

Poisson's Equations:

$$\nabla^2 p \equiv \nabla \cdot \nabla p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f(x, y) \tag{8.1}$$

and the special case of Laplace's equation.

Laplace's Equations:

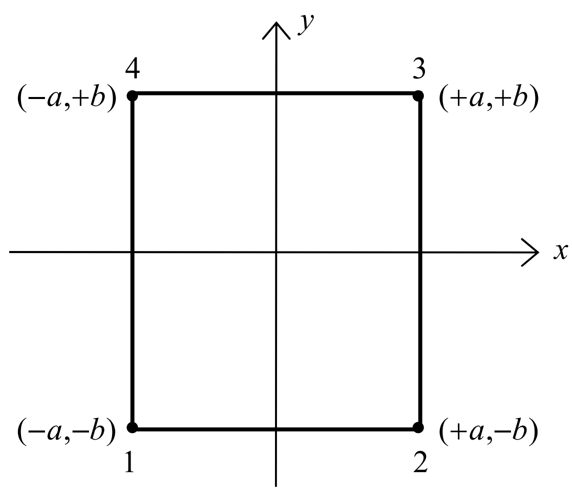
$$\nabla^2 p \equiv \nabla \cdot \nabla p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{8.2}$$

8.1 Solution using Q4 elements

8.1.1 Example 1: Laplace's Equation with One Element

Consider Laplace's equation over the rectangular region $-a \le x \le +a$, $-b \le y \le +b$, with boundary conditions

$$\left.\frac{\partial p}{\partial y}\right|_{(x,-b)} = 1, \qquad p(x,+b) = 0$$
$$\left.\frac{\partial p}{\partial x}\right|_{(-a,y)} = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(+a,y)} = 0$$



The exact solution is the linear function

$p = y - b$ and thus the problem can be solved exactly using just one Q4 element, the one shown here.

The weighted residual integral equation is

$$\iint_S (\nabla \cdot \nabla p)\omega dS = 0 \tag{8.3}$$

and the weak form is (this is one of *Green's identities* – the two dimensional "integration by parts" – see the Appendix to this Chapter)

$$\int_C (\nabla p \cdot \mathbf{n})\omega dC - \iint_S (\nabla p \cdot \nabla \omega) dS = 0 \tag{8.4}$$

Here $\nabla p \cdot \mathbf{n}$ is the normal component of the vector ∇p, and this quantity multiplied by ω is integrated around the boundary C of the element. The normal is taken *outward* from the boundary, so for example if one is integrating along the edge 1-2, $\mathbf{n}$ points in the negative y direction. The notation $\partial p / \partial n$ is often used for this scalar $\nabla p \cdot \mathbf{n}$. Thus

$$\int_{-b}^{+b}\int_{-a}^{+a} (\nabla p \cdot \nabla \omega) dxdy = \int_C \frac{\partial p}{\partial n} \omega dC \tag{8.4}$$

Using the linear interpolation

$$p(x, y) = \sum_{i=1}^{4} p_i N_i \tag{8.5}$$

one has

$$\sum_{i=1}^{4} p_i \int_{-b}^{+b}\int_{-a}^{+a} (\nabla N_i \cdot \nabla N_j) dxdy = \int_C \frac{\partial p}{\partial n} N_j dC, \qquad j = 1 \ldots 4 \tag{8.6}$$

In terms of base vectors $\mathbf{e}_x, \mathbf{e}_y$, the gradients can be written as

$$\begin{aligned}
\nabla N_1 &= \frac{1}{4ab}\left[-(b-y)\mathbf{e}_x - (a-x)\mathbf{e}_y\right] \\
\nabla N_2 &= \frac{1}{4ab}\left[+(b-y)\mathbf{e}_x - (a+x)\mathbf{e}_y\right] \\
\nabla N_3 &= \frac{1}{4ab}\left[+(b+y)\mathbf{e}_x + (a+x)\mathbf{e}_y\right] \\
\nabla N_4 &= \frac{1}{4ab}\left[-(b+y)\mathbf{e}_x + (a-x)\mathbf{e}_y\right]
\end{aligned} \tag{8.7}$$

and the integrals result in a symmetric matrix:

$$\int_{-b}^{+b}\int_{-a}^{+a}\left(\nabla N_i \cdot \nabla N_j\right)dxdy = \mathbf{K}_{\text{rect}} \tag{8.8}$$

where

$$\mathbf{K}_{\text{rect}} = \frac{1}{6ab}\begin{bmatrix} 2a^2+2b^2 & a^2-2b^2 & -a^2-b^2 & -2a^2+b^2 \\ \cdots & 2a^2+2b^2 & -2a^2+b^2 & -a^2-b^2 \\ \cdots & \cdots & 2a^2+2b^2 & a^2-2b^2 \\ \cdots & \cdots & \cdots & 2a^2+2b^2 \end{bmatrix} \tag{8.9}$$

Examining the boundary integrals,

$$\int_C \frac{\partial p}{\partial n}\omega dC \quad \rightarrow \int_{1-2}\frac{\partial p}{\partial n}N_j dC_1 + \int_{2-3}\frac{\partial p}{\partial n}N_j dC_2 + \int_{3-4}\frac{\partial p}{\partial n}N_j dC_3 + \int_{4-1}\frac{\partial p}{\partial n}N_j dC_4 \tag{8.10}$$

where C_1 is the first edge, joining nodes 1 and 2 of the element, etc. As discussed in Chapter 6, two of these integrals on the right hand side will be zero for any given shape function N_j. The right hand side boundary vector is thus

$$\int_C \frac{\partial p}{\partial n} N_j dC = \begin{bmatrix} \int_{1-2} \frac{\partial p}{\partial n} N_1 dC_1 + \int_{4-1} \frac{\partial p}{\partial n} N_1 dC_4 \\ \int_{1-2} \frac{\partial p}{\partial n} N_2 dC_1 + \int_{2-3} \frac{\partial p}{\partial n} N_2 dC_2 \\ \int_{2-3} \frac{\partial p}{\partial n} N_3 dC_2 + \int_{3-4} \frac{\partial p}{\partial n} N_3 dC_3 \\ \int_{3-4} \frac{\partial p}{\partial n} N_4 dC_3 + \int_{4-1} \frac{\partial p}{\partial n} N_4 dC_4 \end{bmatrix} \tag{8.11}$$

Homogeneous natural boundary conditions are applied along 2-3 and 4-1 and so these boundary integrals are zero. For the non-homogeneous natural boundary condition one needs to evaluate

$$\begin{aligned} \int_{1-2} \frac{\partial p}{\partial n} N_1 dC_1 &= \int_{-a}^{+a} \left(-\frac{\partial p}{\partial y} \right) N_1 \Big|_{y=-b} dx = -\frac{1}{2a} \int_{-a}^{+a} (a-x) dx = -a \\ \int_{1-2} \frac{\partial p}{\partial n} N_2 dC_1 &= \int_{-a}^{+a} \left(-\frac{\partial p}{\partial y} \right) N_2 \Big|_{y=-b} dx = -\frac{1}{2a} \int_{-a}^{+a} (a+x) dx = -a \end{aligned} \tag{8.12}$$

leading to the system of equations

$$\mathbf{K}_{\text{rect}} \mathbf{p} = \mathbf{f} \tag{8.13}$$

where $\mathbf{p} = [p_1, p_2, p_3, p_4]^{\mathrm{T}}$, $\mathbf{K}_{\text{rect}}$ is given above, and

$$\mathbf{f} = \begin{bmatrix} -a & -a & \int_{3-4} \frac{\partial p}{\partial n} N_3 dC_3 & \int_{3-4} \frac{\partial p}{\partial n} N_4 dC_3 \end{bmatrix}^{\mathrm{T}} \tag{8.14}$$

Finally, applying the essential boundary condition $p(x,+b) = 0$ leads to

$$\begin{aligned} &\frac{1}{6ab} \begin{bmatrix} 2a^2 + 2b^2 & a^2 - 2b^2 \\ a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -a \\ -a \end{bmatrix} \\ &\rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -2b \\ -2b \end{bmatrix} \end{aligned} \tag{8.15}$$

which is the exact solution. Note that, computationally, one would not reduce the order of the matrix as done above; it is easier to amend the 4×4 matrix to include the essential boundary conditions, as shown here:

$$\frac{1}{6ab}\begin{bmatrix} 2a^2+2b^2 & a^2-2b^2 & 0 & 0 \\ \cdots & 2a^2+2b^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -a \\ -a \\ 0 \\ 0 \end{bmatrix} \tag{8.16}$$

and then to solve. Putting the obtained values of p back into the un-amended system of equations leads to expressions for line integrals along edge 3-4:

$$\begin{aligned} a &= \int_{3-4} \frac{\partial p}{\partial n} N_3 dC_3 = \int_{+a}^{-a} \left.\frac{\partial p}{\partial y}\right|_{3-4} N_3\big|_{y=+b}(-dx) = \frac{1}{2a}\int_{-a}^{+a} \left.\frac{\partial p}{\partial y}\right|_{3-4} (a+x)dx \\ a &= \int_{3-4} \frac{\partial p}{\partial n} N_4 dC_3 = \int_{+a}^{-a} \left.\frac{\partial p}{\partial y}\right|_{3-4} N_4\big|_{y=+b}(-dx) = \frac{1}{2a}\int_{-a}^{+a} \left.\frac{\partial p}{\partial y}\right|_{3-4} (a-x)dx \end{aligned} \tag{8.17}$$

from which one can see that

$$\left.\frac{\partial p}{\partial y}\right|_{3-4} = 1 \tag{8.18}$$

which is again the exact solution.

Note that

$$\nabla p = \sum_{i1}^{4} p_i \nabla N_i = \mathbf{e}_y \tag{8.19}$$

which is also the exact solution.

Different boundary conditions can be taken, for example the set

$$\left.\frac{\partial p}{\partial y}\right|_{(x,-b)} = 0, \qquad \left.\frac{\partial p}{\partial y}\right|_{(x,+b)} = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(-a,y)} = 1, \qquad p(+a,y) = 0 \tag{8.20}$$

has the exact solution $p = x - a$ and the single Q4 element FE equations are, with

$$\begin{aligned} \int_{4-1} \frac{\partial p}{\partial n} N_1 dC_4 &= \int_{+b}^{-b} \left(-\frac{\partial p}{\partial x}\right) N_1\big|_{x=-a} (-dy) = -\frac{1}{2b}\int_{-b}^{+b} (b-y)dy = -b \\ \int_{4-1} \frac{\partial p}{\partial n} N_4 dC_4 &= \int_{+b}^{-b} \left(-\frac{\partial p}{\partial x}\right) N_4\big|_{x=-a} (-dy) = -\frac{1}{2b}\int_{-b}^{+b} (b+y)dy = -b \end{aligned}, \tag{8.21}$$

and

$$\mathbf{K}_{\text{rect}}\mathbf{p} = \mathbf{f}, \qquad \mathbf{f} = \begin{bmatrix} -b & \int_{2-3} \frac{\partial p}{\partial n} N_2 dC_2 & \int_{2-3} \frac{\partial p}{\partial n} N_3 dC_2 & -b \end{bmatrix}^{\mathrm{T}} \tag{8.22}$$

where $\mathbf{K}_{\text{rect}}$ is as before. Applying the essential boundary condition then leads to

$$\begin{aligned} &\frac{1}{6ab}\begin{bmatrix} 2a^2+2b^2 & -2a^2+b \\ -2a^2+b^2 & 2a^2+2b^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_4 \end{bmatrix} = \begin{bmatrix} -b \\ -b \end{bmatrix} \\ &\rightarrow \begin{bmatrix} p_1 \\ p_4 \end{bmatrix} = \begin{bmatrix} -2a \\ -2a \end{bmatrix} \end{aligned} \tag{8.23}$$

which is the exact solution.

8.1.2 Example 2: Poisson's Equation with Local Coordinates

Consider Poisson's equation, with $f = 1$ in (8.1), over the same rectangular region as for Example 1. The weak form of the weighted integral is now

$$\iint_S (\nabla p \cdot \nabla \omega) dS = \int_C (\nabla p \cdot \mathbf{n}) \omega dC - \iint_S \omega dS \tag{8.24}$$

Using the linear interpolation $p(x,y)=\sum_{i=1}^{4} p_i N_i$ one has

$$\sum_{i=1}^{4} p_i \iint_S \left(\nabla N_i \cdot \nabla N_j\right) dS = \int_C \frac{\partial p}{\partial n} N_j \, dC - \iint_S N_j \, dS, \qquad j = 1 \ldots 4 \tag{8.25}$$

Transforming the surface integrals into integrals over the local coordinates, one has

$$\sum_{i=1}^{4} p_i \int_{\eta=-1}^{+1} \int_{\xi=-1}^{+1} \left(\nabla N_i \cdot \nabla N_j\right) J \, d\xi d\eta = \int_C \frac{\partial p}{\partial n} N_j \, dC - \int_{\eta=-1}^{+1} \int_{\xi=-1}^{+1} N_j J \, d\xi d\eta, \qquad j = 1 \ldots 4 \tag{8.26}$$

where J is the Jacobian of the transformation, $J = |\mathbf{J}| = |\mathbf{N}_d \mathbf{x}|$.

To write the integrands in local coordinates, note that

$$\nabla N_i \cdot \nabla N_j = \left(\frac{\partial N_i}{\partial x} \mathbf{e}_x + \frac{\partial N_i}{\partial y} \mathbf{e}_y \right) \cdot \left(\frac{\partial N_j}{\partial x} \mathbf{e}_x + \frac{\partial N_j}{\partial y} \mathbf{e}_y \right) = \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \tag{8.27}$$

When the element edges are aligned with the global x, y axis, these can be evaluated exactly:

$$\nabla N_i \cdot \nabla N_j = \frac{1}{a^2} \frac{\partial N_i}{\partial \xi} \frac{\partial N_j}{\partial \xi} + \frac{1}{b^2} \frac{\partial N_i}{\partial \eta} \frac{\partial N_j}{\partial \eta}$$

$$\to \frac{1}{16} \begin{bmatrix} +\frac{(1-\eta)^2}{a^2} + \frac{(1-\xi)^2}{b^2} & -\frac{(1-\eta)^2}{a^2} + \frac{(1-\xi)(1+\xi)}{b^2} & -\frac{(1-\eta)(1+\eta)}{a^2} - \frac{(1-\xi)(1+\xi)}{b^2} & +\frac{(1-\eta)(1+\eta)}{a^2} - \frac{(1-\xi)^2}{b^2} \\ \cdots & +\frac{(1-\eta)^2}{a^2} + \frac{(1+\xi)^2}{b^2} & +\frac{(1-\eta)(1+\eta)}{a^2} - \frac{(1+\xi)^2}{b^2} & -\frac{(1-\eta)(1+\eta)}{a^2} - \frac{(1-\xi)(1+\xi)}{b^2} \\ \cdots & \cdots & +\frac{(1+\eta)^2}{a^2} + \frac{(1+\xi)^2}{b^2} & -\frac{(1+\eta)^2}{a^2} + \frac{(1-\xi)(1+\xi)}{b^2} \\ \cdots & \cdots & \cdots & +\frac{(1+\eta)^2}{a^2} + \frac{(1-\xi)^2}{b^2} \end{bmatrix} \tag{8.28}$$

Carrying out the integration leads to

$$\int_{\eta=-1}^{+1}\int_{\xi=-1}^{+1}\left(\nabla N_i \cdot \nabla N_j\right) J\, d\xi d\eta = \mathbf{K}_{\text{rect}}, \qquad \int_{\eta=-1}^{+1}\int_{\xi=-1}^{+1} N_j J\, d\xi d\eta = ab\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \tag{8.29}$$

where $\mathbf{K}_{\text{rect}}$ is the same coefficient matrix as encountered in Example 1.

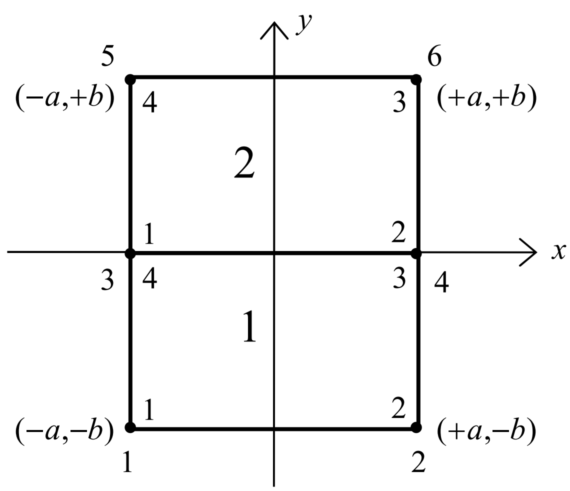
8.1.3 Example 3: A Two-element Problem

As a further example, consider again the Poisson's equation of Example 2 over the rectangular region $-a \le x \le +a$, $-b \le y \le +b$, with boundary conditions

$$\left.\frac{\partial p}{\partial y}\right|_{(x,-b)} = 1, \qquad p(x,+b) = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(-a,y)} = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(+a,y)} = 0 \tag{8.30}$$

The exact solution is the quadratic function $p = \frac{1}{2}y^2 + (1+b)y - b(1+\frac{3}{2}b)$.

The problem will be solved using one element spanning the x direction and two elements spanning the y direction, as shown.



The complete symmetric system of equations for this two-element mesh is (note that in what follows, the b, the element "half-height" is now replaced with $\bar{b} = b/2$)

$$\mathbf{Kp} = \mathbf{f}$$

where $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix}^{\text{T}}$ and

$$\mathbf{K} = \frac{1}{6a\bar{b}}\begin{bmatrix} 2a^2+2\bar{b}^2 & a^2-2\bar{b}^2 & -2a^2+\bar{b}^2 & -a^2-\bar{b}^2 & 0 & 0 \\ \cdots & 2a^2+2\bar{b}^2 & -a^2-\bar{b}^2 & -2a^2+\bar{b}^2 & 0 & 0 \\ \cdots & \cdots & 4a^2+4\bar{b}^2 & 2a^2-4\bar{b}^2 & -2a^2+\bar{b}^2 & -a^2-\bar{b}^2 \\ \cdots & \cdots & \cdots & 4a^2+4\bar{b}^2 & -a^2-\bar{b}^2 & -2a^2+\bar{b}^2 \\ \cdots & \cdots & \cdots & \cdots & 2a^2+2\bar{b}^2 & a^2-2\bar{b}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 2a^2+2\bar{b}^2 \end{bmatrix} \tag{8.31}$$

$$\mathbf{f} = \begin{bmatrix} \left(\int_{1-2}\frac{\partial p}{\partial n}N_1 dC_1 + \int_{4-1}\frac{\partial p}{\partial n}N_1 dC_4\right)_{\text{elem 1}} \\ \left(\int_{1-2}\frac{\partial p}{\partial n}N_2 dC_1 + \int_{2-3}\frac{\partial p}{\partial n}N_2 dC_2\right)_{\text{elem 1}} \\ \left(\int_{3-4}\frac{\partial p}{\partial n}N_4 dC_3 + \int_{4-1}\frac{\partial p}{\partial n}N_4 dC_4\right)_{\text{elem 1}} + \left(\int_{1-2}\frac{\partial p}{\partial n}N_1 dC_1 + \int_{4-1}\frac{\partial p}{\partial n}N_1 dC_4\right)_{\text{elem 2}} \\ \left(\int_{2-3}\frac{\partial p}{\partial n}N_3 dC_2 + \int_{3-4}\frac{\partial p}{\partial n}N_3 dC_3\right)_{\text{elem 1}} + \left(\int_{1-2}\frac{\partial p}{\partial n}N_2 dC_1 + \int_{2-3}\frac{\partial p}{\partial n}N_2 dC_2\right)_{\text{elem 2}} \\ \left(\int_{2-3}\frac{\partial p}{\partial n}N_3 dC_2 + \int_{3-4}\frac{\partial p}{\partial n}N_3 dC_3\right)_{\text{elem 2}} \\ \left(\int_{3-4}\frac{\partial p}{\partial n}N_4 dC_3 + \int_{4-1}\frac{\partial p}{\partial n}N_4 dC_4\right)_{\text{elem 2}} \end{bmatrix} - a\bar{b}\begin{bmatrix}1\\1\\2\\2\\1\\1\end{bmatrix} \tag{8.32}$$

From the above, with the normal derivatives constant, the boundary integrals for both elements are given by

$$\int_{1-2} N_j dC_1 = \begin{bmatrix}a\\a\\0\\0\end{bmatrix}, \quad \int_{2-3} N_j dC_2 = \begin{bmatrix}0\\\bar{b}\\\bar{b}\\0\end{bmatrix}, \quad \int_{3-4} N_j dC_3 = \begin{bmatrix}0\\0\\a\\a\end{bmatrix}, \quad \int_{4-1} N_j dC_4 = \begin{bmatrix}\bar{b}\\0\\0\\\bar{b}\end{bmatrix} \tag{8.33}$$

The homogeneous natural boundary conditions imply that the line integrals along edges 2-3 and 4-1 are zero for both elements. The non-homogeneous natural boundary condition, $\partial p/\partial y|_{(x,-b)} = 1$, so $\partial p/\partial n|_{1-2} = -1$, leads to the right hand side vector $[-a \quad -a \quad 0 \quad 0]^{\mathrm{T}}$.

Note that the integrals over the internal edge cancel out:

$$\left(\int_{3-4}\frac{\partial p}{\partial n}N_j dC_3\right)_{\text{elem}\,1}+\left(\int_{1-2}\frac{\partial p}{\partial n}N_j dC_1\right)_{\text{elem}\,2}=\left(\left.\frac{\partial p}{\partial n}\right|_{3-4}\right)_{\text{elem}\,1}\begin{bmatrix}0\\0\\a\\a\\0\\0\end{bmatrix}+\left(\left.\frac{\partial p}{\partial n}\right|_{1-2}\right)_{\text{elem}\,2}\begin{bmatrix}0\\0\\a\\a\\0\\0\end{bmatrix}=0 \quad (8.34)$$

The remaining boundary integral leads to

$$\left(\left.\frac{\partial p}{\partial n}\right|_{3-4}\right)_{\text{elem}\,2}\begin{bmatrix}0\\0\\a\\a\end{bmatrix} \quad (8.35)$$

Thus

$$\mathbf{f}=\begin{bmatrix}-a\\-a\\0\\0\\a\left(\left.\frac{\partial p}{\partial n}\right|_{3-4}\right)_{\text{elem}\,2}\\a\left(\left.\frac{\partial p}{\partial n}\right|_{3-4}\right)_{\text{elem}\,2}\end{bmatrix}-a\overline{b}\begin{bmatrix}1\\1\\2\\2\\1\\1\end{bmatrix} \quad (8.36)$$

Applying the essential boundary conditions finally leads to the system

$$\frac{1}{6a\overline{b}}\begin{bmatrix}2a^2+2\overline{b}^2 & a^2-2\overline{b}^2 & -2a^2+\overline{b}^2 & -a^2-\overline{b}^2\\ \cdots & 2a^2+2\overline{b}^2 & -a^2-\overline{b}^2 & -2a^2+\overline{b}^2\\ \cdots & \cdots & 4a^2+4\overline{b}^2 & 2a^2-4\overline{b}^2\\ \cdots & \cdots & \cdots & 4a^2+4\overline{b}^2\end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\\p_4\end{bmatrix}=\begin{bmatrix}-a\\-a\\0\\0\end{bmatrix}-a\overline{b}\begin{bmatrix}1\\1\\2\\2\end{bmatrix} \quad (8.37)$$

which has the solution

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -2b(1+b) \\ -2b(1+b) \\ -b(1+3b/2) \\ -b(1+3b/2) \end{bmatrix} \tag{8.38}$$

which are the exact values at the nodes (although the FE solution is linear between the nodes and the exact solution is quadratic).

With the p's now known, the normal derivative between nodes 5 and 6 can be determined, giving

$$\left(\left. \frac{\partial p}{\partial n} \right|_{3-4} \right)_{\text{elem } 2} = \left. \frac{\partial p}{\partial y} \right|_{y=+b} = 1 + 2b \tag{8.39}$$

which is also the exact solution.

The exact gradient is linear: $\nabla p = (1 + b + y)\mathbf{e}_y$ The FE gradients are constant (the subscripts on the p's here are local element numbering)

$$\begin{aligned} \nabla p &= \sum_{i=1}^{4} p_i \nabla N_i \\ &= \sum_{i=1}^{4} p_i \left(\frac{1}{a} \frac{\partial N_i}{\partial \xi} \mathbf{e}_x + \frac{1}{b} \frac{\partial N_i}{\partial \eta} \mathbf{e}_y \right) \end{aligned}$$

Element 1:

$$\begin{aligned} &= \frac{1}{4a} \left[p_1(-1+\eta) + p_2(+1-\eta) + p_3(+1+\eta) + p_4(-1-\eta) \right] \mathbf{e}_x \\ &\qquad + \frac{1}{4b} \left[p_1(-1+\xi) + p_2(-1-\xi) + p_3(+1+\xi) + p_4(+1-\xi) \right] \mathbf{e}_y \\ &= \left[1 + \frac{1}{2} b \right] \mathbf{e}_y \end{aligned}$$

$$
\begin{aligned}
\nabla p &= \sum_{i=1}^{4} p_i \nabla N_i \\
&= \sum_{i=1}^{4} p_i \left(\frac{1}{a}\frac{\partial N_i}{\partial \xi}\mathbf{e}_x + \frac{1}{\overline{b}}\frac{\partial N_i}{\partial \eta}\mathbf{e}_y \right) \\
\text{Element 2:} \quad &= \frac{1}{4a}\left[p_1(-1+\eta) + p_2(+1-\eta) + p_3(+1+\eta) + p_4(-1-\eta)\right]\mathbf{e}_x \\
&\quad + \frac{1}{4\overline{b}}\left[p_1(-1+\xi) + p_2(-1-\xi) + p_3(+1+\xi) + p_4(+1-\xi)\right]\mathbf{e}_y \\
&= \left[1 + \frac{3}{2}b\right]\mathbf{e}_y
\end{aligned}
$$

These gradients are the average of the exact linear gradients over each element, they are exact at the element-centres and they are not continuous across the element boundary.

8.2 Solution using Rotated Nonconforming Q1 elements

First, re-consider the example 1 with exact solution $p = y - b$. Again using one element with the element edges aligned with the global x, y axis, one has

$$
\int_{\eta=-1}^{+1}\int_{\xi=-1}^{+1} \left(\nabla N_i \cdot \nabla N_j\right) J \, d\xi d\eta = \mathbf{K}_{\text{RQ\_rect}} \tag{8.40}
$$

where now the symmetric coefficient matrix is

$$
\mathbf{K}_{\text{RQ\_rect}} = \frac{1}{28ab}\begin{bmatrix} +37a^2+65b^2 & -37a^2-37b^2 & +37a^2+9b^2 & -37a^2-37b^2 \\ & +65a^2+37b^2 & -37a^2-37b^2 & +9a^2+37b^2 \\ & & +37a^2+65b^2 & -37a^2-37b^2 \\ & & & +65a^2+37b^2 \end{bmatrix} \tag{8.41}
$$

leading to the system of equations $\mathbf{K}_{\text{rect}}\mathbf{p} = \mathbf{f}$ where $\mathbf{p} = [p_1, p_2, p_3, p_4]^{\mathrm{T}}$ and

$$
\mathbf{f} = \left[0 \quad +2a\left.\frac{\partial p}{\partial n}\right|_{C_2} \quad 0 \quad -2a \right]^{\mathrm{T}} \tag{8.42}
$$

Finally, applying the essential boundary condition $p(x,+b) = 0$ leads to

$$\frac{1}{28ab}\begin{bmatrix} +37a^2+65b^2 & +37a^2+9b^2 & -37a^2-37b^2 \\ +37a^2+9b^2 & +37a^2+65b^2 & -37a^2-37b^2 \\ -37a^2-37b^2 & -37a^2-37b^2 & +65a^2+37b^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2a \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} p_1 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -b \\ -b \\ -2b \end{bmatrix} \tag{8.43}$$

which is the exact solution. The solution is exact between the nodes; although the interpolation functions are not linear, due to cancellation of terms one arrives at

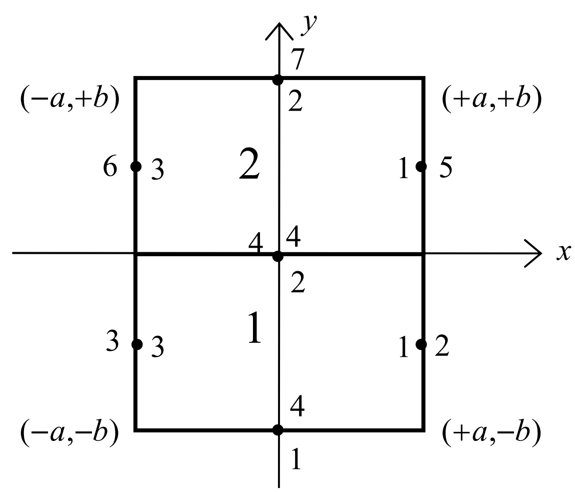
$$p(0,y) = \sum_{i=1}^{4} p_i N_i = -b\left[N_1 + N_3 + 2N_4\right] = b - y \quad \text{for any } x \tag{8.44}$$

Putting the solution back into the original system of equations, from the second equation one finds that $\partial p/\partial n\big|_{C_2} = 1$, which is the exact solution. Finally,

$$\nabla p = \sum_{i=1}^{4} p_i \nabla N_i = \sum_{i=1}^{4} p_i \left(\frac{1}{a}\frac{\partial N_i}{\partial \xi}\mathbf{e}_x + \frac{1}{b}\frac{\partial N_i}{\partial \eta}\mathbf{e}_y \right) = \mathbf{e}_y \tag{8.45}$$

which is also exact.

Next look at the two-element problem of example 3, with exact solution $p = \frac{1}{2}y^2 + (1+b)y - b(1+\frac{3}{2}b)$. The mesh is now as shown.



The symmetric system of equations for this two-element mesh is now (again $\bar{b} = b/2$) $\mathbf{Kp} = \mathbf{f}$ Where $\mathbf{p} = \left[p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7\right]^{\mathrm{T}}$ and

$$
\mathbf{K} = \frac{1}{28a\bar{b}}
\begin{bmatrix}
65a^2 + 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 & +9a^2 + 37\bar{b}^2 & 0 & 0 & 0 \\
\cdots & +37a^2 + 65\bar{b}^2 & +37a^2 + 9\bar{b}^2 & -37a^2 - 37\bar{b}^2 & 0 & 0 & 0 \\
\cdots & \cdots & +37a^2 + 65\bar{b}^2 & -37a^2 - 37\bar{b}^2 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & +130a^2 + 74\bar{b}^2 & -37a^2 - 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 & +9a^2 + 37\bar{b}^2 \\
\cdots & \cdots & \cdots & \cdots & +37a^2 + 65\bar{b}^2 & +37a^2 + 9\bar{b}^2 & -37a^2 - 37\bar{b}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & +37a^2 + 65\bar{b}^2 & -37a^2 - 37\bar{b}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & +65a^2 + 37\bar{b}^2
\end{bmatrix}
\tag{8.46}
$$

and

$$
\mathbf{f} = \begin{bmatrix} -2a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2a(\partial p / \partial y)_7 \end{bmatrix} - a\bar{b} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\tag{8.47}
$$

Applying the essential boundary condition $p_7 = 0$ finally leads to the system

$$
\frac{1}{28a\bar{b}}
\begin{bmatrix}
65a^2 + 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 & +9a^2 + 37\bar{b}^2 & 0 & 0 \\
\cdots & +37a^2 + 65\bar{b}^2 & +37a^2 + 9\bar{b}^2 & -37a^2 - 37\bar{b}^2 & 0 & 0 \\
\cdots & \cdots & +37a^2 + 65\bar{b}^2 & -37a^2 - 37\bar{b}^2 & 0 & 0 \\
\cdots & \cdots & \cdots & +130a^2 + 74\bar{b}^2 & -37a^2 - 37\bar{b}^2 & -37a^2 - 37\bar{b}^2 \\
\cdots & \cdots & \cdots & \cdots & +37a^2 + 65\bar{b}^2 & +37a^2 + 9\bar{b}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & +37a^2 + 65\bar{b}^2
\end{bmatrix}
\times \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}
= \begin{bmatrix} -2a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - a\bar{b} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}
\tag{8.48}
$$

which has the solution (compared to the exact solution)

$$\text{FE: } \begin{bmatrix} p_1 \\ p_2 = p_3 \\ p_4 \\ p_5 = p_6 \end{bmatrix} = \begin{bmatrix} -2b(1+b) \\ -b\left(\dfrac{3}{2} + \dfrac{7}{4} b \dfrac{\frac{156}{37} a^2 + b^2}{4a^2 + b^2}\right) \\ -b(1+3b/2) \\ -b\left(\dfrac{1}{2} + \dfrac{3}{4} b \dfrac{\frac{500}{111} a^2 + b^2}{4a^2 + b^2}\right) \end{bmatrix}, \quad \text{Exact: } \begin{bmatrix} p_1 \\ p_2 = p_3 \\ p_4 \\ p_5 = p_6 \end{bmatrix} = \begin{bmatrix} -2b(1+b) \\ -b\left(\dfrac{3}{2} + \dfrac{15}{8} b\right) \\ -b(1+3b/2) \\ -b\left(\dfrac{1}{2} + \dfrac{7}{8} b\right) \end{bmatrix} \tag{8.49}$$

With the p's now known, the normal derivative between nodes 5 and 6 can be determined, giving

$$\left(\left. \frac{\partial p}{\partial n} \right|_{C_2} \right)_{\text{elem 2}} = \left. \frac{\partial p}{\partial y} \right|_{y=+b} = 1 + 2b \tag{8.50}$$

which is the exact solution (even though the inexact p_5 and p_6 are used to calculate it).

The exact gradient is linear: $\nabla p = (1 + b + y)\mathbf{e}_y$ The FE gradients are

Element 1:

$$\begin{aligned} \nabla p &= \sum_{i=1}^{4} p_i \nabla N_i \\ &= \sum_{i=1}^{4} p_i \left(\frac{1}{a} \frac{\partial N_i}{\partial \xi} \mathbf{e}_x + \frac{1}{\overline{b}} \frac{\partial N_i}{\partial \eta} \mathbf{e}_y \right) \\ &= \left[\frac{7}{37} \frac{ab^2}{4a^2 + b^2} \left(3\xi - 10\xi^3 \right) \right] \mathbf{e}_x + \left[-\frac{14}{37} \frac{a^2 b}{4a^2 + b^2} \left(3\eta - 10\eta^3 \right) + 1 + \frac{1}{2} b \right] \mathbf{e}_y \end{aligned}$$

Element 2:

$$\begin{aligned} \nabla p &= \sum_{i=1}^{4} p_i \nabla N_i \\ &= \sum_{i=1}^{4} p_i \left(\frac{1}{a} \frac{\partial N_i}{\partial \xi} \mathbf{e}_x + \frac{1}{\overline{b}} \frac{\partial N_i}{\partial \eta} \mathbf{e}_y \right) \\ &= \left[\frac{7}{37} \frac{ab^2}{4a^2 + b^2} \left(3\xi - 10\xi^3 \right) \right] \mathbf{e}_x + \left[-\frac{14}{37} \frac{a^2 b}{4a^2 + b^2} \left(3\eta - 10\eta^3 \right) + 1 + \frac{3}{2} b \right] \mathbf{e}_y \end{aligned}$$

These gradients are exact at the element-centres and they are not continuous across the element boundary.

8.3 Solution using Linear Triangular Elements

8.3.1 Example 4: Laplace's Equation

Consider Laplace's equation, as in Example 1 only now over the rectangular region $0 \le x \le a$, $0 \le x \le b$, with boundary conditions

$$\left.\frac{\partial p}{\partial y}\right|_{(x,0)} = 1, \qquad p(x,b) = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(0,y)} = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(a,y)} = 0 \tag{8.51}$$

whose exact solution is again the linear function $p = y - b$, with $\partial p / \partial y = 1$. Using the linear interpolation $\sum_{i=1}^{3} p_i N_i$ in the weighted integral (8.4) leads to

$$2\Delta \sum_{i=1}^{3} p_i \int_{\eta=0}^{1} \int_{\xi=0}^{1-\xi} \left(\nabla N_i \cdot \nabla N_j\right) d\eta d\xi = \int_C \frac{\partial p}{\partial n} \omega dC \tag{8.52}$$

Since the elements are linear, this integrand is constant for any given element:

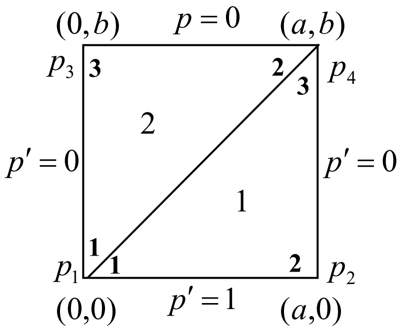
$$\begin{aligned}\nabla N_i \cdot \nabla N_j &= \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \\ &= \frac{1}{(2\Delta)^2}\left\{\begin{bmatrix} y_2 - y_3 \\ y_3 - y_1 \\ y_1 - y_2 \end{bmatrix}\begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix} + \begin{bmatrix} x_2 - x_3 \\ x_3 - x_1 \\ x_1 - x_2 \end{bmatrix}\begin{bmatrix} x_2 - x_3 & x_3 - x_1 & x_1 - x_2 \end{bmatrix}\right\}\end{aligned} \tag{8.53}$$

Using the two elements shown below, with $\int_{\eta=0}^{1}\int_{\xi=0}^{1-\xi} d\eta d\xi = 1/2$, the local element matrices are

$$\mathbf{K}_{\text{elem}\,1} = \frac{1}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2 + b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix}, \quad \mathbf{K}_{\text{elem}\,2} = \frac{1}{2ab}\begin{bmatrix} a^2 & 0 & -a^2 \\ 0 & b^2 & -b^2 \\ -a^2 & -b^2 & a^2 + b^2 \end{bmatrix} \tag{8.54}$$

and the assembled global matrix is

$$\mathbf{K} = \frac{1}{2ab}\begin{bmatrix} a^2+b^2 & -b^2 & -a^2 & 0 \\ -b^2 & a^2+b^2 & 0 & -a^2 \\ -a^2 & 0 & a^2+b^2 & -b^2 \\ 0 & -a^2 & -b^2 & a^2+b^2 \end{bmatrix} \tag{8.55}$$



The natural boundary condition leads to the right hand side vector

$$\mathbf{f} = -\frac{a}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \frac{a}{2}\left.\frac{\partial p}{\partial y}\right|_{(x,b)}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} \tag{8.56}$$

Applying the essential boundary condition, $p_3 = p_4 = 0$, leads to the solution $p_1 = p_2 = -b$ and $(\partial p/\partial y)_{(x,b)} = 1$ which is the exact solution.

Consider now the evaluation of the gradient ∇p throughout both elements. One has (the subscripts are now local node numbers)

Element 1: $\nabla p = \dfrac{p_2 - p_1}{a}\mathbf{e}_x + \dfrac{p_3 - p_2}{b}\mathbf{e}_y = \mathbf{e}_y$

Element 2: $\nabla p = \dfrac{p_2 - p_3}{a}\mathbf{e}_x + \dfrac{p_3 - p_1}{b}\mathbf{e}_y = \mathbf{e}_y$

Which is the exact solution. The gradient is a constant vector and, in this simple example, the gradient is continuous across the element boundary.

8.3.2 Example 5: Poisson's Eqn.

Consider here Poisson's equation over the rectangular region $0 \le x \le a,\ 0 \le y \le b$. The element equations are

$$\sum p_i \int_S (\nabla N_i)\cdot(\nabla N_j)dS = -\int_E fN_j dS + \int_{C_E} (\nabla p \cdot \mathbf{n})N_j dC \tag{8.57}$$

Constant *f*

For the case of constant *f*, the equations become

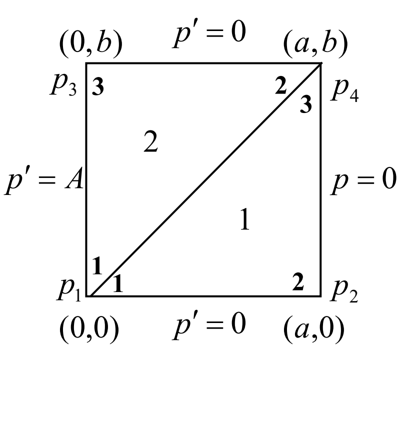
$$\sum p_i \int_S (\nabla N_i)\cdot(\nabla N_j)dS = -f\frac{\Delta}{3} + \int_{C_E} (\nabla p \cdot \mathbf{n})N_j dC \tag{8.58}$$

Consider the boundary conditions

$$\left.\frac{\partial p}{\partial y}\right|_{(x,0)} = 0, \qquad \left.\frac{\partial p}{\partial y}\right|_{(x,b)} = 0, \qquad \left.\frac{\partial p}{\partial x}\right|_{(0,y)} = A, \qquad p(a,y) = 0 \tag{8.59}$$

which give the exact solution (a quadratic in *x*)

$$p(x,y) = -\frac{1}{2}f\left(a^2 - x^2\right) - A(a-x), \qquad p'(x,y) = fx + A \tag{8.60}$$



Using the same two-element mesh as in Example 4, and so with the same element and global matrices, but with the right-hand side vector

$$\mathbf{f} = -f\frac{\Delta}{3}\begin{bmatrix}2\\1\\1\\2\end{bmatrix} - \frac{b}{2}A\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \frac{b}{2}\left.\frac{\partial p}{\partial x}\right|_{(a,y)}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \tag{8.61}$$

and with $p_2 = p_4 = 0$, the pressures can be obtained:

$$p_1 = -\frac{1}{2} fa^2 \frac{1+\frac{2}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2} - Aa, \quad p_3 = -\frac{1}{2} fa^2 \frac{1+\frac{1}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2} - Aa \qquad (8.62)$$

which approaches the exact solution as $b/a \to 0$; the average pressure along the left-hand edge is, though, the exact solution $p(0,y) = -fa^2/2$. The 2nd and 4th equations yield different values for the right-hand edge flux, but their average is the exact solution, $fa + A$.

The exact gradient ∇p is linear: $(fx + A)\mathbf{e}_x$. The FE solution is constant within each element,

$$\text{Element 1:} \qquad \nabla p = \frac{p_2 - p_1}{a}\mathbf{e}_x + \frac{p_3 - p_2}{b}\mathbf{e}_y = \left[\frac{1}{2} fa \frac{1+\frac{2}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2} + A\right]\mathbf{e}_x$$

$$\text{Element 2:} \qquad \begin{aligned} \nabla p &= \frac{p_2 - p_3}{a}\mathbf{e}_x + \frac{p_3 - p_1}{b}\mathbf{e}_y \\ &= \left[\frac{1}{2} fa \frac{1+\frac{1}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2} + A\right]\mathbf{e}_x + \left[\frac{1}{2} fa^2 \frac{1}{b}\frac{\frac{1}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2}\right]\mathbf{e}_y \end{aligned}$$

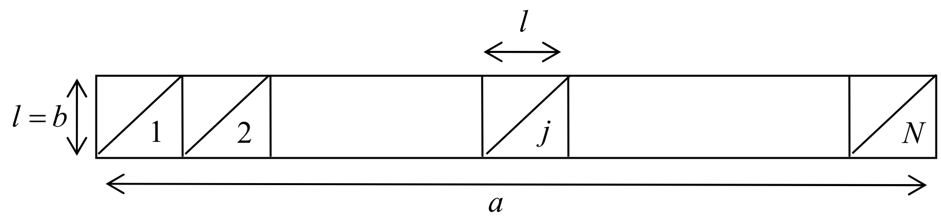
The component of ∇p normal to the edge common to both elements can be obtained by taking the dot product of these gradients with the normal vector $\mathbf{n} = (b\mathbf{e}_x - a\mathbf{e}_y)/\sqrt{a^2 + b^2}$, leading to

$$\text{Element 1:} \qquad \nabla p \cdot \mathbf{n} = \frac{b}{\sqrt{a^2 + b^2}}\left[\frac{1}{2} fa \frac{1+\frac{2}{3}(b/a)^2}{1+\frac{1}{2}(b/a)^2} + A\right]$$

$$\text{Element 2:} \qquad \nabla p \cdot \mathbf{n} = \frac{b}{\sqrt{a^2 + b^2}}\left[\frac{1}{2} fa \frac{\frac{2}{3}+\frac{1}{2}(b/a)^2}{1+\frac{1}{2}(b/a)^2} + A\right]$$

showing that the normal component is not continuous across the element boundary, for any value of b/a. The exact solution at the centre of the square is $\nabla p \cdot \mathbf{n} = b\left[\frac{1}{2} fa + A\right]/\sqrt{a^2 + b^2}$.

Suppose now that one has a long mesh of $2N$ triangles, with each "box" of equal height and width l, equal to b and a/N respectively, as illustrated.



With this mesh, the FE solution converges to the exact solution as *N* increases. For example, with $N=1$ (the example considered above), the FE solution at $x=0$ is $-0.444fa^2 - aA$ (upper node) and $-0.556fa^2 - aA$ (lower node); for $N=4$, the solution is $-0.496fa^2 - aA$ (upper node) and $-0.504fa^2 - aA$ (lower node); again, the average is the exact solution.

Non-constant *f*

Consider the case of varying *f*, in particular the problem with $f = x$, subject to the same boundary conditions (8.59), which has the exact solution

$$p(x,y) = \frac{1}{6}\left(x^3 - a^3\right) - A(a-x), \qquad p'(x,y) = \frac{1}{2}x^2 + A \tag{8.63}$$

In this problem one needs to evaluate

$$\int_E fN_j dS = -\int_E xN_j dS = -\sum_{i=1}^{3} x_i \int_E N_i N_j dS = \frac{\Delta}{12}\left\{ -x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \tag{8.64}$$

Using the same two-element mesh as in Example 4, the right hand side vector becomes

$$\mathbf{f} = -\frac{a^2 b}{24}\begin{bmatrix} 3 \\ 3 \\ 1 \\ 5 \end{bmatrix} - \frac{b}{2}A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{b}{2}\left.\frac{\partial p}{\partial x}\right|_{(a,y)}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{8.65}$$

With $p_2 = p_4 = 0$, one obtains:

$$p_1 = -\frac{a^3}{6}\frac{1+\frac{3}{4}(b/a)^2}{1+\frac{1}{2}(b/a)^2} - Aa, \quad p_3 = -\frac{a^3}{6}\frac{1+\frac{1}{4}(b/a)^2}{1+\frac{1}{2}(b/a)^2} - Aa \tag{8.66}$$

The average value of p along the left-hand edge is the exact solution $p(0,y) = -a^3/6$. The 2nd and 4th equations yield different values for the right-hand edge flux, but their average is the exact solution, $a^2/2 + A$.

Again, it can be shown that the component of ∇p normal to the edge common to both elements is not continuous across that boundary.

Finally, consider the problem with $f(x,y) = 2\left(x^2 + y^2\right)$ subject to

$$p(0,y) = 0, \qquad p(a,y) = a^2 y^2, \qquad p(x,0) = 0, \qquad p(x,b) = b^2 x^2 \tag{8.67}$$

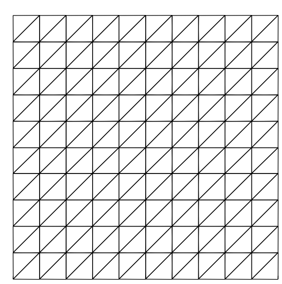
which has the exact solution

$$p(x,y) = x^2 y^2, \quad \frac{\partial p}{\partial x} = 2xy^2, \quad \frac{\partial p}{\partial y} = 2x^2 y \tag{8.68}$$

This problem involves the evaluation of the integrals

$$\int_E f N_j dS = -2\int_E \left(x^2 + y^2\right) N_j dS = -2\int_E \left(\left(\sum_{i=1}^{3} x_i N_i\right)^2 + \left(\sum_{i=1}^{3} y_i N_i\right)^2\right) N_j dS \tag{8.69}$$

The problem was solved for $a = b = 2$, with the square region divided into N right-sided triangles of equal size. The pressure at the mid-point is shown in the table below for different mesh refinements (the 200 element mesh is shown).



N	$p(1,1)$
8	0.83333

32	0.95313
72	0.97863
128	0.98787
200	0.99220
Exact	**1.00000**

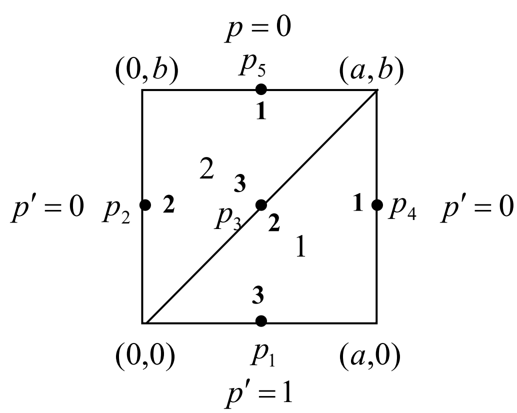
8.4 Solution using Nonconforming Triangular Element

8.4.1 Example 6: Laplace's Eqn.

Consider again Example 4, Laplace's equation with the boundary conditions (8.51). With the non-conforming element, because of the different Jacobian, (8.52) now reads

$$8\Delta_s \sum_{i=1}^{3} p_i \int_{\eta=0}^{1} \int_{\xi=0}^{1-\xi} \left(\nabla N_i \cdot \nabla N_j\right) d\eta d\xi = \int_C \frac{\partial p}{\partial n} \omega dC \tag{8.70}$$

Again, the integrand is a constant for any given element and is given by (8.53), only with the x_i, y_i now being the non-conforming (mid-side) nodal coordinates and Δ is replaced by Δ_s. Using the two elements shown, with $\int_{\eta=0}^{1}\int_{\xi=0}^{1-\xi} d\eta d\xi = 1/2$, the local element matrices are



$$\mathbf{K}_{\text{elem}\,1} = \frac{2}{ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix},$$

$$\mathbf{K}_{\text{elem}\,2} = \frac{2}{ab}\begin{bmatrix} a^2 & 0 & -a^2 \\ 0 & b^2 & -b^2 \\ -a^2 & -b^2 & a^2+b^2 \end{bmatrix} \tag{8.71}$$

and the assembled global matrix is

$$\mathbf{K} = \frac{2}{ab}\begin{bmatrix} a^2 & 0 & -a^2 & 0 & 0 \\ 0 & b^2 & -b^2 & 0 & 0 \\ -a^2 & -b^2 & 2a^2+2b^2 & -b^2 & -a^2 \\ 0 & 0 & -b^2 & b^2 & 0 \\ 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix} \tag{8.72}$$

The natural boundary condition leads to the right hand side vector

$$\mathbf{f} = -a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a\left.\frac{\partial p}{\partial y}\right|_{(x,b)}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{8.73}$$

Applying the essential boundary condition, $p_5 = 0$ then leads to the solution $p_1 = -b,\ p_2 = p_3 = p_4 = -b/2$. The final equation then gives $(\partial p/\partial y)_{(x,b)} = 1$.

The gradient is (the subscripts here are local node numbers)

Element 1: $\nabla p = \frac{p_1 - p_2}{a/2}\mathbf{e}_x + \frac{p_2 - p_3}{b/2}\mathbf{e}_y = \mathbf{e}_y$

Element 2: $\nabla p = \frac{p_3 - p_2}{a/2}\mathbf{e}_x + \frac{p_1 - p_3}{b/2}\mathbf{e}_y = \mathbf{e}_y$

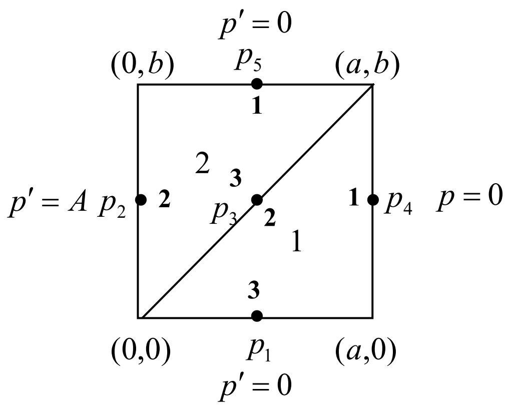
Again, in this simple example, the gradient is continuous across the element boundary.

8.4.2 Example 7: Poisson’s Eqn.

Consider again Poisson’s equation over the rectangular region $0 \le x \le a,\ 0 \le y \le b$.

Constant *f*

Consider again the problem with constant *f* considered in §8.3.2, Example 5. Using the same two-element mesh as in Example 6, shown here, and so with the same element and global matrices, but with the right-hand side vector



$$\mathbf{f} = -f\frac{ab}{6}\begin{bmatrix}1\\1\\2\\1\\1\end{bmatrix} - bA\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + b\left.\frac{\partial p}{\partial x}\right|_{(a,y)}\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \quad (8.74)$$

and using $p_4 = 0$, leads to the exact pressure at node 2. The FE solution is (compared with the exact solution)

$$\text{FE:}\begin{bmatrix}p_1\\p_2\\p_3\\p_5\end{bmatrix} = \begin{bmatrix}-\frac{5}{12}fa^2 - \frac{1}{12}fb^2 - \frac{1}{2}Aa\\ -\frac{1}{2}fa^2 - Aa\\ -\frac{5}{12}fa^2 - \frac{1}{2}Aa\\ -\frac{5}{12}fa^2 - \frac{1}{12}fb^2 - \frac{1}{2}Aa\end{bmatrix}, \quad \text{Exact:}\begin{bmatrix}p_1\\p_2\\p_3\\p_5\end{bmatrix} = \begin{bmatrix}-\frac{3}{8}fa^2 - \frac{1}{2}Aa\\ -\frac{1}{2}fa^2 - Aa\\ -\frac{3}{8}fa^2 - \frac{1}{2}Aa\\ -\frac{3}{8}fa^2 - \frac{1}{2}Aa\end{bmatrix} \quad (8.75)$$

Using (the inexact) p_3, the 4th equation yields the exact solution for the flux at node 4, $fa + A$.

The exact gradient ∇p is linear: $(fx + A)\mathbf{e}_x$. The FE solution is constant within each element (the subscripts here are local node numbers)

$$\text{Element 1:} \quad \nabla p = \frac{p_1 - p_2}{a/2}\mathbf{e}_x + \frac{p_2 - p_3}{b/2}\mathbf{e}_y = \left[\frac{5}{6}fa + A\right]\mathbf{e}_x + \left[+\frac{1}{6}fb\right]\mathbf{e}_y$$

$$\text{Element 2:} \quad \nabla p = \frac{p_3 - p_2}{a/2}\mathbf{e}_x + \frac{p_1 - p_3}{b/2}\mathbf{e}_y = \left[\frac{1}{6}fa + A\right]\mathbf{e}_x + \left[-\frac{1}{6}fb\right]\mathbf{e}_y$$

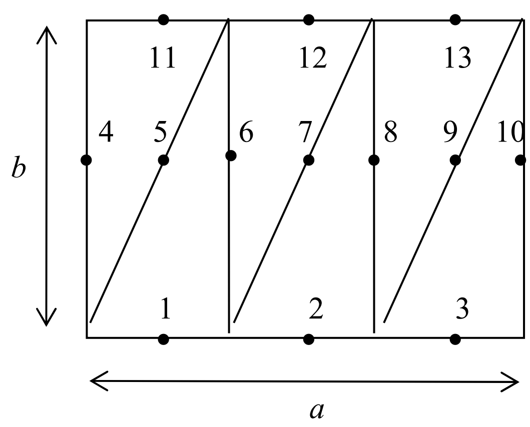
The component of ∇p normal to the edge common to both elements can be obtained by taking the dot product of these gradients with the normal vector $\mathbf{n} = (b\mathbf{e}_x - a\mathbf{e}_y)/\sqrt{a^2+b^2}$, leading to

Element 1: $$\nabla p \cdot \mathbf{n} = \frac{b}{\sqrt{a^2+b^2}}\left[\frac{2}{3}fa + A\right]$$

Element 2: $$\nabla p \cdot \mathbf{n} = \frac{b}{\sqrt{a^2+b^2}}\left[\frac{1}{3}fa + A\right]$$

showing that the normal component is not continuous across the element boundary, for any value of b/a. The exact solution at the centre of the square is $\nabla p \cdot \mathbf{n} = b\left[\frac{1}{2}fa + A\right]/\sqrt{a^2+b^2}$, which is the average of the two FE values.

The solution converges to the exact solution as the mesh is made finer. For example, the FE solution for the centre of the region is $-0.3750 fa^2 - \frac{1}{2}Aa$. With the above two-element mesh, the FE solution is $-0.4167 fa^2 - \frac{1}{2}Aa$. With the 6-element mesh shown below it is $-0.3796 fa^2 - \frac{1}{2}Aa$ (at node 7). The solution is exact at nodes 4, 6, 8 and 10.



The full solution for this mesh is

$$\text{FE:}\begin{bmatrix} p_1 = p_{11} \\ p_2 = p_{12} \\ p_3 = p_{13} \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{bmatrix} = \begin{bmatrix} -\frac{53}{108} fa^2 - \frac{1}{12} fb^2 - \frac{5}{6} Aa \\ -\frac{41}{108} fa^2 - \frac{1}{12} fb^2 - \frac{1}{2} Aa \\ -\frac{17}{108} fa^2 - \frac{1}{12} fb^2 - \frac{1}{6} Aa \\ -\frac{1}{2} fa^2 - Aa \\ -\frac{53}{108} fa^2 - \frac{5}{6} Aa \\ -\frac{4}{9} fa^2 - \frac{2}{3} Aa \\ -\frac{41}{108} fa^2 - \frac{1}{2} Aa \\ -\frac{5}{18} fa^2 - \frac{1}{3} Aa \\ -\frac{17}{108} fa^2 - \frac{1}{6} Aa \end{bmatrix}, \ \text{Exact:}\begin{bmatrix} p_1 = p_{11} \\ p_2 = p_{12} \\ p_3 = p_{13} \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} fa^2 - \frac{5}{6} Aa \\ -\frac{3}{8} fa^2 - \frac{1}{2} Aa \\ -\frac{11}{72} fa^2 - \frac{1}{6} Aa \\ -\frac{1}{2} fa^2 - Aa \\ -\frac{5}{12} fa^2 - \frac{5}{6} Aa \\ -\frac{4}{9} fa^2 - \frac{2}{3} Aa \\ -\frac{3}{8} fa^2 - \frac{1}{2} Aa \\ -\frac{5}{18} fa^2 - \frac{1}{3} Aa \\ -\frac{11}{72} fa^2 - \frac{1}{6} Aa \end{bmatrix}$$

8.5 Appendix to Chapter 8

8.5.1 Green's Identity for two dimensional integrals

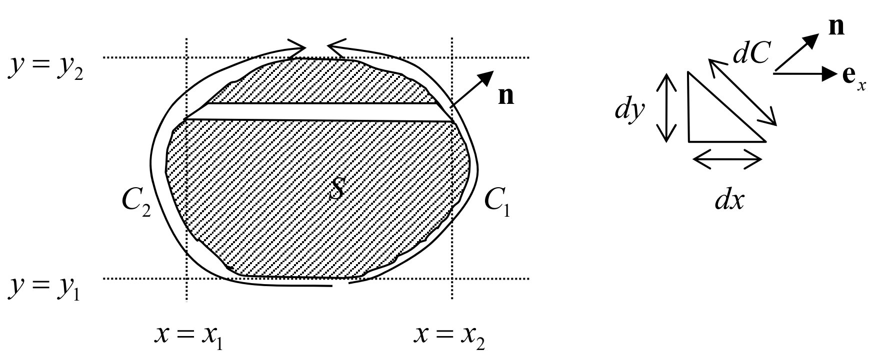
Consider the weighted residual integral equation

$$\iint_S (\nabla \cdot \nabla p)\omega dS = \iint_S \left(\omega \frac{\partial^2 p}{\partial x^2} + \omega \frac{\partial^2 p}{\partial y^2} \right) dS = 0 \tag{8A.1}$$

Consider the first term and write it as

$$\int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} \omega \frac{\partial^2 p}{\partial x^2} dx \right) dy = 0 \tag{8A.2}$$

Here, y_1, y_2 are the minimum and maximum values of y in the domain S and x_1, x_2 are the minimum and maximum values of x for the strip of height dy as it moves from y_1 to y_2, as shown in the figure below.



Integrating by parts with respect to x leads to

$$\int_{y_1}^{y_2}\left(\int_{x_1}^{x_2}\omega\frac{\partial^2 p}{\partial x^2}dx\right)dy = \int_{y_1}^{y_2}\left(\left[\omega\frac{\partial p}{\partial x}\right]_{x_1}^{x_2}\right)dy - \int_{y_1}^{y_1}\left(\int_{x_1}^{x_2}\frac{\partial \omega}{\partial x}\frac{\partial p}{\partial x}dx\right)dy$$
$$= \int_{y_1}^{y_2}\left(\left[\omega\frac{\partial p}{\partial x}\right]_{x_1}^{x_2}\right)dy - \iint_S \frac{\partial \omega}{\partial x}\frac{\partial p}{\partial x}dS \quad (8A.3)$$

The first term can be re-written in terms of normals in the x and y directions, as indicated in the figure:

$$dy = \cos\theta dC = (\mathbf{n}\cdot\mathbf{e}_x)dC = n_x dC \quad (8A.4)$$

where n_x is the component of the outward normal $\mathbf{n}$ in the x direction, so that

$$\int_{y_1}^{y_2}\left(\int_{x_1}^{x_2}\omega\frac{\partial^2 p}{\partial x^2}dx\right)dy = \int_{C_1}\omega\frac{\partial p}{\partial x}n_x C_1 - \int_{C_2}\omega\frac{\partial p}{\partial x}n_x C_2 - \iint_S \frac{\partial \omega}{\partial x}\frac{\partial p}{\partial x}dS$$
$$= \int_C \omega\frac{\partial p}{\partial x}n_x dC - \iint_S \frac{\partial \omega}{\partial x}\frac{\partial p}{\partial x}dS \quad (8A.5)$$

where C is the complete boundary, and C_1, C_2 are as shown in the figure.

Similarly, the second term, involving y, can be expressed as

$$\int_S \omega\frac{\partial^2 p}{\partial y^2}dS = \int_C \omega\frac{\partial p}{\partial y}n_y dC - \iint_S \frac{\partial \omega}{\partial y}\frac{\partial p}{\partial y}dS \quad (8A.6)$$

Combining both parts leads to

$$\int_S \omega\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)dS = \int_C \omega\left(\frac{\partial p}{\partial x}n_x + \frac{\partial p}{\partial y}n_y\right)dC - \iint_S\left(\frac{\partial \omega}{\partial x}\frac{\partial p}{\partial x}+\frac{\partial \omega}{\partial y}\frac{\partial p}{\partial y}\right)dS \quad (8A.7)$$

This can be written in vector notation:

$$\int_S (\nabla \cdot \nabla p)\omega dS = \int_C \omega \left[\left(\frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y \right) \cdot \mathbf{n} \right] dC - \iint_S \left[\left(\frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y \right) \cdot \left(\frac{\partial w}{\partial x}\mathbf{e}_x + \frac{\partial w}{\partial y}\mathbf{e}_y \right) \right] dS$$

$$= \int_C (\nabla p \cdot \mathbf{n}) \omega dC - \iint_S (\nabla p \cdot \nabla \omega) dS \qquad \text{(8A.8)}$$