3 NonStandard Galerkin FEM

The standard Galerkin FEM as described in chapter 2 is a powerful tool for the numerical solution of a wide variety of problems. However, there are certain problems which cannot be solved with adequate accuracy using the standard GFEM, for example the analysis of nearly-incompressible materials or convection/diffusion with large Reynolds numbers; new variants of the standard FEM have been proposed to deal with problems of this type. Other types of GFEM have been proposed as attractive alternatives to the standard strategies. As an introduction to these variants of the FEM, below are discussed briefly the **Penalty Method**, some **Mixed Methods** and a **Non-Mixed Conservative Method**. These methods are considered "advanced" and do not need to be studied on a first reading.

3.1 The Penalty Method

The Penalty Method involves a very simple idea – essential boundary conditions do not have to be *strongly* enforced, but can be imposed *weakly*.

It has been seen that to apply an essential boundary condition, to node 1 say, $p_1 = \overline{p}_1$, one can alter the FE system as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & K_{22} & K_{23} & \cdots & K_{2n} \\ 0 & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \overline{p}_1 \\ f_2 - K_{21}\overline{p}_1 \\ f_3 - K_{31}\overline{p}_1 \\ \vdots \\ f_n - K_{n1}\overline{p}_1 \end{bmatrix} \tag{3.1}$$

This is the strong method of applying the boundary condition, building it into the system. An alternative method is to impose it weakly, as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} + \eta \begin{bmatrix} p_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} + \eta \begin{bmatrix} \bar{p}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.2}$$

Here, the term(s) involving η is called a **penalty** term; the penalty η must be large enough to drive p_1 to $\bar{p}_1$, but not so large as to cause numerical problems.

The method does not hold much of an advantage over the standard FEM, but it is worth studying because it forms the basis of more powerful Galerkin FEMs, the **Internal Penalty** (IP) method and the **Discontinuous** Galerkin FEM. In these latter methods, the trial polynomials *p* over each element may be discontinuous at common nodes – they are forced to be continuous by penalty.

More formally, consider the following problem with non-homogeneous essential boundary conditions:

$$\frac{\partial^2 p}{\partial x^2} + f(x) = 0, \qquad p(0) = p(1) = \bar{p} \tag{3.3}$$

The weak formulation including the penalty term is

$$\int \frac{\partial p}{\partial x}\frac{\partial \omega}{\partial x}dx - \left[\frac{\partial p}{\partial x}\omega\right] + \left[\eta(p - \bar{p})\omega\right] = \int f\omega dx \tag{3.4}$$

For a mesh of $n-1$ linear elements of equal length *L*, the two integrals lead to the standard global system

$$\frac{1}{L}\begin{bmatrix} +1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ & & -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ f_n \end{bmatrix} \tag{3.5}$$

The two boundary terms are treated as follows (this formulation relies on the fact that the shape functions take the values 0 or 1 at a boundary node):

$$\left[\frac{\partial p}{\partial x}\omega\right] = \frac{1}{L}(p_{i+1} - p_i)[N_j] \rightarrow \frac{1}{L}\begin{bmatrix} +1 & -1 & & \\ & & & \\ & & -1 & +1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$$+\eta[(p-\overline{p})N_j] \rightarrow \eta\begin{bmatrix} -1 & & \\ & & \\ & & +1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \eta\begin{bmatrix} -\overline{p} \\ \\ +\overline{p} \end{bmatrix} \tag{3.6}$$

leading to the final system

$$\left\{\frac{1}{L}\begin{bmatrix} +1 & 0 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ & & 0 & +1 \end{bmatrix} + \eta\begin{bmatrix} -1 & & \\ & & \\ & & +1 \end{bmatrix}\right\}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} + \eta\begin{bmatrix} -\overline{p} \\ 0 \\ \vdots \\ +\overline{p} \end{bmatrix} \tag{3.7}$$

Taking now $\eta = C/L$, where C is a constant, it can be seen that

$$p_1 = -\frac{L}{C+1}f_1 + \frac{C}{C+1}\overline{p}, \qquad p_n = +\frac{L}{C+1}f_n + \frac{C}{C+1}\overline{p} \tag{3.8}$$

and sufficient accuracy is obtained by choosing C to be sufficiently large.

3.1.1 Symmetric Systems

The system can be made symmetric by including an extra boundary term (it can be included because $p \rightarrow 0$ at the boundary):

$$\int \frac{\partial p}{\partial x}\frac{\partial \omega}{\partial x}dx - \left[\frac{\partial p}{\partial x}\omega\right] - \left[\frac{\partial \omega}{\partial x}p\right] + [\eta\omega p] = \int f\omega dx \tag{3.9}$$

This results in the final system {▲Problem 1}

$$\left\{ \frac{1}{L}\begin{bmatrix} +1 & 0 & & \\ 0 & 2 & -1 & \\ & -1 & \ddots & 0 \\ & & 0 & +1 \end{bmatrix} + \eta \begin{bmatrix} -1 & & \\ & & \\ & & +1 \end{bmatrix} \right\} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} + \eta \begin{bmatrix} -\bar{p} \\ 0 \\ \vdots \\ +\bar{p} \end{bmatrix} - \frac{\bar{p}}{L} \begin{bmatrix} +1 \\ -1 \\ \vdots \\ -1 \\ +1 \end{bmatrix} \tag{3.10}$$

For example, taking $f(x) = x$ (in which case the exact solution is $p = \bar{p} + x(1-x^2)/6$), the f vector is

$$\frac{L^2}{4}\begin{bmatrix} A_1 - 1/3 \\ A_1 + A_2 \\ A_2 + A_3 \\ \vdots \\ A_{n-1} + 1/3 \end{bmatrix}, \qquad A_i = \left(\frac{x_{i+1} + x_i}{x_{i+1} - x_i} \right) \tag{3.11}$$

The value of p at the right hand side, for $\bar{p} = 1$ and 5 elements, is then as shown in the table below (convergence at the left hand end is much better for a given C).

C	$p(1)$
1	0.009333333
10	0.819878788
100	0.980382838
1000	0.998020646
10000	0.999801886

3.1.2 Natural Boundary Conditions

Natural boundary conditions can be treated as in the standard FEM. For example, consider next the following problem:

$$\frac{\partial^2 p}{\partial x^2} = A, \qquad p(0) = \bar{p}(0), \quad p'(1) = \bar{p}'(1) \tag{3.12}$$

[exact solution: $p(x) = \frac{1}{2}Ax^2 + (\bar{p}'(1) - A)x + \bar{p}(0)$]

In this case one arrives at the system

$$\left\{ \frac{1}{L} \begin{bmatrix} +1 & 0 & & \\ 0 & 2 & -1 & \\ & -1 & \ddots & -1 \\ & & -1 & +1 \end{bmatrix} + \eta \begin{bmatrix} -1 & & \\ & & \\ & & \\ & & 0 \end{bmatrix} \right\} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ +\overline{p}'(1) \end{bmatrix} + \eta \begin{bmatrix} -\overline{p}(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \frac{\overline{p}(0)}{L} \begin{bmatrix} +1 \\ -1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{3.13}$$

3.2 The Mixed Method

It often happens that the most important variable to be solved for is not the independent variable *p*, but its derivative $q = \partial p / \partial x$. It has been seen that, with the standard Galerkin FEM, the solution for *q* is of an order less accurate than the solution for *p*. In mixed methods, equations are set up for the solution of *p* and *q* simultaneously (as with the cubic Hermite element). Different interpolation schemes (weight functions) can be used for *p* and *q*, depending on the accuracy required. In the most basic case, *q* is interpolated linearly between nodes whilst *p* is constant over an element (reversing the accuracy obtained with the standard FEM). Obtaining a more accurate solution for the derivative takes some more computational effort than obtaining a sufficiently accurate *p*.

3.2.1 The Standard Mixed Method

The standard Mixed Method will be illustrated by solving the problem

$$\frac{\partial^2 p}{\partial x^2} = -1 \text{ subject to } q(0) = 1, \quad p(2) = 0, \text{ where } q = \frac{\partial p}{\partial x}$$

[exact solution: $x - \frac{1}{2}x^2$]

(3.14)

Solution I (standard Galerkin FEM)

First, recall the standard FEM: one writes

$$
\int_{x_i}^{x_{i+1}} \frac{\partial^2 p}{\partial x^2} w dx = -\int_{x_i}^{x_{i+1}} w dx
$$
$$
\rightarrow \quad p_i \frac{1}{L}\begin{bmatrix} +1 \\ -1 \end{bmatrix} + p_{i+1}\frac{1}{L}\begin{bmatrix} -1 \\ +1 \end{bmatrix} = \begin{bmatrix} -p'(x_i) \\ +p'(x_{i+1}) \end{bmatrix} + \frac{L}{2}\begin{bmatrix} +1 \\ +1 \end{bmatrix} \tag{3.15}
$$

Two elements, with $L = 1$, give the exact pressures at the nodes as plotted below left, $\begin{bmatrix} p_1 & p_2 \end{bmatrix}^T = \begin{bmatrix} 0 & 1/2 \end{bmatrix}^T$. The system of equations to solve is $(N+1)\times(N+1)$ for N elements, but a little additional work needs to be done to evaluate the derivative q. For the two elements, the derivatives are $q = \partial p/\partial x = (p_{i+1} - p_i)/L$, so

element 1: $q = +\frac{1}{2}$

element 2: $q = -\frac{1}{2}$

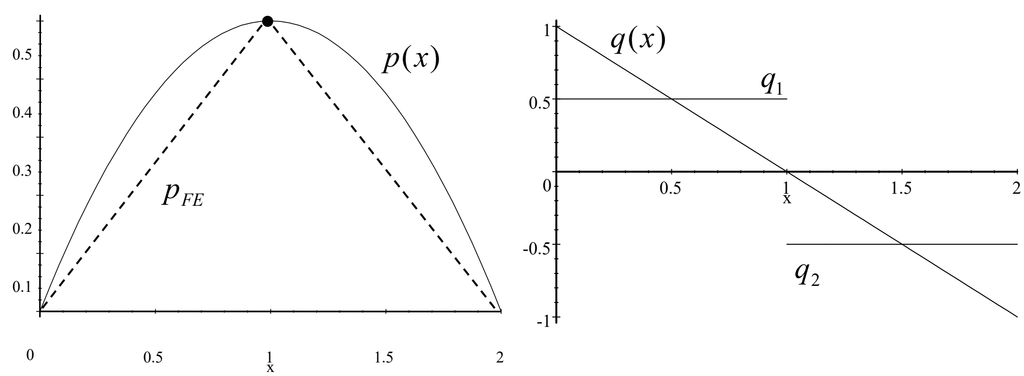

Figure 3.1: Standard GFEM solution to Eqn. 3.14

Solution II (Standard Mixed FEM)

In the (standard) Mixed Method, one can take q to vary linearly over an element and p to be constant over an element, and replace the second order Eqn. 3.14 by the two separate first order equations:

$$\frac{\partial q}{\partial x} = -1, \qquad q = \frac{\partial p}{\partial x} \tag{3.16}$$

This allows one to solve for p and q simultaneously. Further, the first of these equations, the **conservation equation**, so called because it often arises in practical problems as an expression of conservation of some property such as mass, will now hold over an element, and this is often important from a physical point of view – it will be noted that, in the standard FEM with linear elements, q is a constant and its derivative is zero; thus this conservation condition is satisfied using the standard FEM *only* in the special case that the governing equation is homogeneous, i.e. $\partial^2 p / \partial x^2 = 0$.

The equations are now discretised in the usual way, with $q = N_i q_i + N_{i+1} q_{i+1}$, Fig. 3.2.

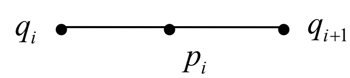

Figure 3.2: Element with *p* constant, *q* varying linearly

The equations on the left here have a constant weight function z (=1), equivalent to a finite difference scheme, those on the right have the standard linear shape/weight functions:

$$\frac{\partial q}{\partial x} = -1 \qquad\qquad q = \frac{\partial p}{\partial x}$$

$$\int_{x_i}^{x_{i+1}} \frac{\partial q}{\partial x} z dx = -\int_{x_i}^{x_{i+1}} z dx \qquad\qquad \int_{x_i}^{x_{i+1}} q w dx = \int_{x_i}^{x_{i+1}} \frac{\partial p}{\partial x} w dx$$

$$q_i \int_{x_i}^{x_{i+1}} \frac{\partial N_1}{\partial x} dx + q_{i+1} \int_{x_i}^{x_{i+1}} \frac{\partial N_2}{\partial x} dx = -(x_{i+1} - x_i)$$

$$q_i \left[\int_{-1}^{+1} \frac{\partial N_1}{\partial \xi} d\xi \right] + q_{i+1} \left[\int_{-1}^{+1} \frac{\partial N_2}{\partial \xi} d\xi \right] = -(x_{i+1} - x_i)$$

$$q_i[-1] + q_{i+1}[+1] = -L$$

$$q_i \int_{x_i}^{x_{i+1}} N_1 N_j dx + q_{i+1} \int_{x_i}^{x_{i+1}} N_2 N_j dx = \left[p N_j \right]_{x_i}^{x_{i+1}} - p_i \int_{x_i}^{x_{i+1}} \frac{\partial N_j}{\partial x} dx$$

$$q_i \left[\frac{L}{2} \int_{-1}^{+1} N_1 N_j d\xi \right] + q_{i+1} \left[\frac{L}{2} \int_{-1}^{+1} N_2 N_j d\xi \right] = \left[p N_j \right]_{x_i}^{x_{i+1}} - p_i \int_{x_i}^{x_{i+1}} \frac{\partial N_j}{\partial x} dx$$

$$q_i \begin{bmatrix} L/3 \\ L/6 \end{bmatrix} + q_{i+1} \begin{bmatrix} L/6 \\ L/3 \end{bmatrix} + p_i \begin{bmatrix} -1 \\ +1 \end{bmatrix} = \begin{bmatrix} -p(x_i) \\ +p(x_{i+!}) \end{bmatrix}$$

(3.17)

Using a single element then gives

$$\begin{bmatrix} L/3 & L/6 & -1 \\ L/6 & L/3 & +1 \\ -1 & +1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} -p(x_1) \\ +p(x_2) \\ -L \end{bmatrix} \rightarrow \begin{bmatrix} L/3 & +1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} q_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} -L/6 \\ -L+1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, \quad p_1 = \frac{1}{3} \tag{3.18}$$

This is the exact solution for q (since it is linear). Note that the coefficient matrix is symmetric. With two elements, one arrives at {▲Problem 2} $p_1 = p_2 = 1/3$ and with four elements one obtains a better solution for the p: $p = \begin{bmatrix} 0.2083 & 0.4583 & 0.4583 & 0.2083 \end{bmatrix}^T$.

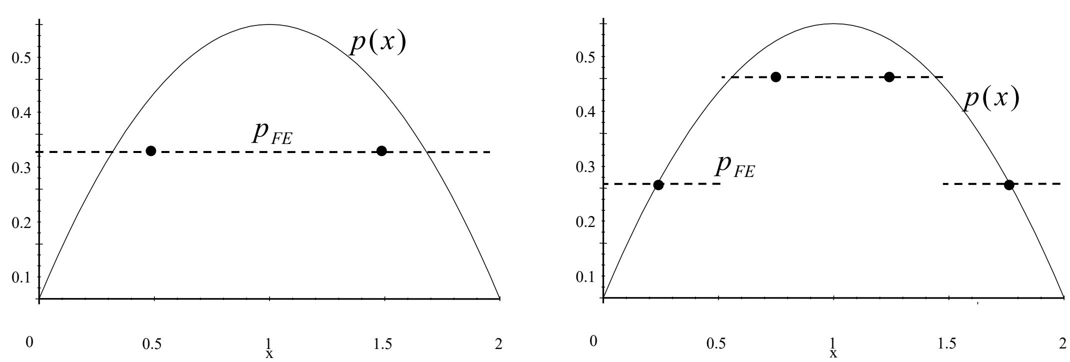

Figure 3.3: Standard Mixed Method solution to Eqn. 3.14 (2 & 4 elements)

Consider now a problem with a cubic solution:

$$\frac{\partial^2 p}{\partial x^2} = x \text{ subject to } p(0) = 2, \quad p(2) = 0\text{, where } q = \frac{\partial p}{\partial x} \tag{3.19}$$

[exact solution: $2 - \frac{5}{3}x + \frac{1}{6}x^3$]

Using the mixed method with two elements leads to {▲Problem 3}

$$\begin{bmatrix} L/3 & L/6 & 0 & -1 & 0 \\ L/6 & 2L/3 & L/6 & +1 & -1 \\ 0 & L/6 & L/3 & 0 & +1 \\ -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -p(x_1) \\ 0 \\ +p(x_{i+1}) \\ L(x_i + L/2) \\ L(x_i + L/2) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -7/4 \\ -5/4 \\ +1/4 \end{bmatrix}, \qquad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1.2083 \\ 0.1250 \end{bmatrix} \tag{3.20}$$

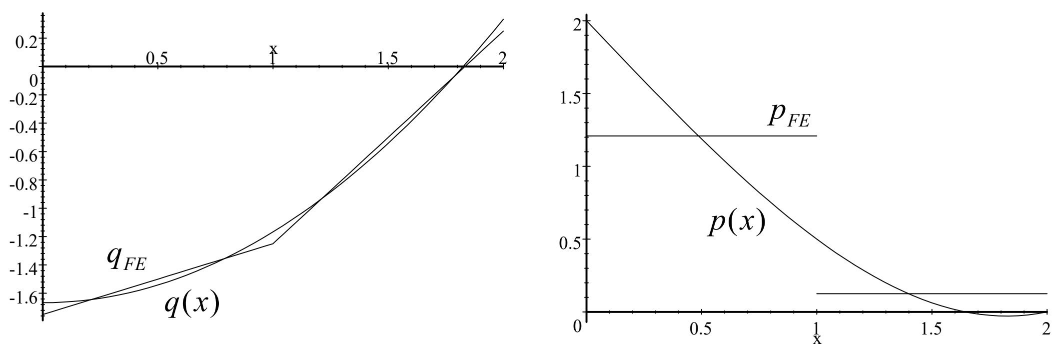

Figure 3.4: Standard Mixed Method solution to Eqn. 3.20

Note that the FE solution here ensures that q is *continuous* across element boundaries.

Higher Order Accuracy

One cannot increase the accuracy of p by letting it vary also linearly over an element, with two unknowns at the node points (element end-points). This is because one cannot have unknown

p's and q's at the *same* node – otherwise the resulting coefficient matrix will be singular, even after application of the boundary conditions.

One cannot circumvent this by having the unknown p's at two interior nodes, i.e. not at the end-points, since then there will be more unknowns than equations.

3.3 Mixed Finite Volume / Covolume Methods

Control Volume methods are widely used in the numerical solution of flow problems. These types of problem have traditionally been solved using the Finite Difference method, but new FEM methods, such as the one described here, are now also being used.

Here problem (3.14) is re-visited using a so-called Control Volume Mixed Finite Element Method[1]. Shown in Fig. 3.5 are three "control volumes" Q_i, $Q_{i+1/2}$ and Q_{i+1}. One can consider Q_i and Q_{i+1} to be "elements" of a primary mesh/grid. The functions p and q are interpolated over these elements. A secondary or dual grid consists of the overlapping volumes $Q_{i+1/2}$, etc.

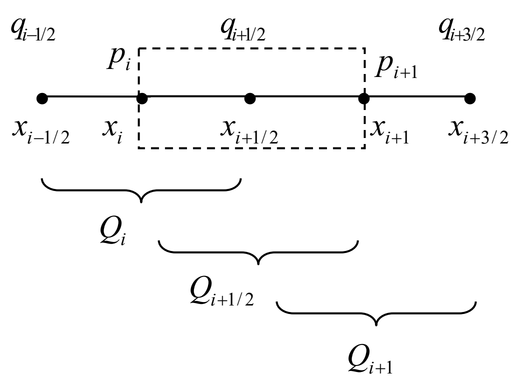

Figure 3.5: Dual grid and Control Volumes

First, consider the equation $q = \dfrac{\partial p}{\partial x}$; integrating over the control volume $Q_{i+1/2}$ gives

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[1] *ref*: Cai Z, Jones JE, McCormick SF, Russell TF, "control-volume mixed finite element methods", Computational Geosciences, 1997;1:289-315

$$\int_{x_i}^{x_{i+1}} q dx - \int_{x_i}^{x_{i+1}} \frac{\partial p}{\partial x} dx = 0$$

$$\rightarrow \int_{x_i}^{x_{i+1}} q dx - \left[p(x_{i+1}) - p(x_i) \right] = 0$$

$$\rightarrow q_{i-1/2} \int_{x_i}^{x_{i+1/2}} N_1 dx + q_{i+1/2} \int_{x_i}^{x_{i+1/2}} N_2 dx + q_{i+1/2} \int_{x_{i+1/2}}^{x_{i+1}} N_1 dx + q_{i+3/2} \int_{x_{i+1/2}}^{x_{i+1}} N_2 dx - \left[p_{i+1} - p_i \right] = 0$$

$$\rightarrow q_{i-1/2} \int_{x_i}^{x_{i+1/2}} \left(1 - \frac{x - x_{i-1/2}}{L_i} \right) dx + q_{i+1/2} \int_{x_i}^{x_{i+1/2}} \left(\frac{x - x_{i-1/2}}{L_i} \right) dx$$

$$+ q_{i+1/2} \int_{x_{i+1/2}}^{x_{i+1}} \left(1 - \frac{x - x_{i+1/2}}{L_{i+1}} \right) dx + q_{i+3/2} \int_{x_{i+1/2}}^{x_{i+1}} \left(\frac{x - x_{i+1/2}}{L_{i+1}} \right) dx - \left[p_{i+1} - p_i \right] = 0$$

$$\rightarrow q_{i-1/2} \left[\frac{L_i}{8} \right] + q_{i+1/2} \left[\frac{3L_i}{8} \right] + q_{i+1/2} \left[\frac{3L_{i+1}}{8} \right] + q_{i+3/2} \left[\frac{L_{i+1}}{8} \right] - \left[p_{i+1} - p_i \right] = 0$$

(3.21)

Next consider the conservation equation $\partial q / \partial x = -1$: as with the standard mixed method, integrating over the elements Q_i and Q_{i+1} gives

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial q}{\partial x} dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} dx \qquad \int_{x_{i+1/2}}^{x_{i+3/2}} \frac{\partial q}{\partial x} dx = -\int_{x_{i+1/2}}^{x_{i+3/2}} dx$$
$$\rightarrow q_{i+1/2} - q_{i-1/2} = -L_i \qquad \rightarrow q_{i+3/2} - q_{i+/2} = -L_{i+1} \tag{3.22}$$

For these two elements under consideration, integrating the equation $q = \partial p / \partial x$ over the two half-sized control volumes at either end, which involve pressure values at the boundaries, give rise to

$$
\begin{aligned}
&\int_{x_{i-1/2}}^{x_i} q\,dx - \int_{x_{i-1/2}}^{x_i} \frac{\partial p}{\partial x} dx = 0 \\
&\quad \to q_{i-1/2} \int_{x_{i-1/2}}^{x_i} N_1 dx + q_{i+1/2} \int_{x_{i-1/2}}^{x_i} N_2 dx - p_i = -p(x_{i-1/2}) \\
&\quad \to q_{i-1/2}\left[\frac{3L_i}{8}\right] + q_{i+1/2}\left[\frac{L_i}{8}\right] - p_i = -p(x_{i-1/2}) \\
&\int_{x_{i+1}}^{x_{i+3/2}} q\,dx - \int_{x_{i+1}}^{x_{i+3/2}} \frac{\partial p}{\partial x} dx = 0 \\
&\quad \to q_{i+1/2} \int_{x_{i+1}}^{x_{i+3/2}} N_1 dx + q_{i+3/2} \int_{x_{i+1}}^{x_{i+3/2}} N_2 dx + p_{i+1} = +p(x_{i-3/2}) \\
&\quad \to q_{i+1}\left[\frac{L_{i+1}}{8}\right] + q_{i+3/2}\left[\frac{3L_{i+1}}{8}\right] + p_{i+1} = +p(x_{i-3/2})
\end{aligned}
\tag{3.23}
$$

The (symmetric) system of equations for two elements is then

$$
\begin{bmatrix}
\frac{3L_i}{8} & \frac{L_i}{8} & 0 & -1 & 0 \\
\frac{L_i}{8} & \frac{3(L_i + L_{i+1})}{8} & \frac{L_{i+1}}{8} & +1 & -1 \\
0 & \frac{L_{i+1}}{8} & \frac{3L_{i+1}}{8} & 0 & +1 \\
-1 & +1 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
q_{i-1/2} \\ q_{i+1/2} \\ q_{i+3/2} \\ p_i \\ p_{i+1}
\end{bmatrix}
=
\begin{bmatrix}
-p(x_{i-1/2}) \\ 0 \\ +p(x_{i-3/2}) \\ -L_i \\ -L_{i+1}
\end{bmatrix}
\tag{3.24}
$$

With $L_i = L_{i+1}$ these are (compare with the slightly different system of equations resulting from the standard mixed method, Eqn 3.20)

$$\begin{bmatrix} 3L/8 & L/8 & 0 & -1 & 0 \\ L/8 & 3L/4 & L/8 & +1 & -1 \\ 0 & L/8 & 3L/8 & 0 & +1 \\ -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_{i-1/2} \\ q_{i+1/2} \\ q_{i+3/2} \\ p_i \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} -p(x_{i-1/2}) \\ 0 \\ +p(x_{i-3/2}) \\ -L \\ -L \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ L/8 & 3L/4 & L/8 & +1 & -1 \\ 0 & L/8 & 3L/8 & 0 & +1 \\ -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_{i-1/2} \\ q_{i+1/2} \\ q_{i+3/2} \\ p_i \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -L \\ -L \end{bmatrix} \rightarrow \begin{bmatrix} q_{i-1/2} \\ q_{i+1/2} \\ q_{i+3/2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} p_i \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} 3/8 \\ 3/8 \end{bmatrix} \tag{3.25}$$

which is similar to the solution from the standard mixed method.

3.4 Non-Mixed Method for Conservative Elements

As mentioned, in the standard FEM the conservation relation $q' = f(x)$ (for example $f(x) = -1$ in problem 1 above) does not usually hold over an element. The mixed methods discussed above ensure that this relation does hold, but the variable *p* is not evaluated accurately.

In the method outlined here, both *p* and *q* are evaluated accurately. Further, control volumes are not used, so all calculations are carried out for each element – there is only one grid.

First[2], suppose that p_h, the FE approximation to *p*, is evaluated using any method, for example the standard GFEM. One can then take a Taylor's series of q_h about the centre of the element x_c:

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[2] *ref*: Chou S-H, Tang S, "Conservative *p1* conforming and non-conforming galerkin FEMs: effective flux evaluation via a nonmixed method approach", SIAM J. Numeric. Anal., 2000;38(2):660-680

$$\begin{aligned} q_h(x) &= q_h(x_c) + (x - x_c)\left.\frac{\partial q}{\partial x}\right|_{x=x_c} \\ &= q_h(x_c) + (x - x_c) f(x_c) \\ &= \frac{\partial p_h}{\partial x} + (x - x_c) f(x_c) \end{aligned} \tag{3.26}$$

In other words, q is evaluated by taking the derivative of p, as in the standard GFEM, and then by adding a *correction* term to make it linear over the element. The accuracy of this depends on how accurate $f(x_c)$ is evaluated, and on how accurate is $\partial q_h(x)/\partial x = f(x_c)$ in any element. There are a number of different ways of evaluating $f(x_c)$, e.g. taking the average of f over an element or using various interpolation schemes (see below). If q is linear, so that $q'' = 0$, etc., and $q'(x_c)$ is evaluated exactly, then $q_h(x)$ thus evaluated will be exact.

Consider the following problem:

$$\frac{\partial q}{\partial x} = 12x^2 \text{ subject to } p(0) = 2, \quad p(2) = 0 \text{, where } q = \frac{\partial p}{\partial x} \tag{3.27}$$

$$\text{[exact solution: } \begin{aligned} p(x) &= 2 - 9x + x^4 \\ q(x) &= -9 + 4x^3 \end{aligned} \text{]}$$

First evaluate p using the standard GFEM. Note that in practical codes, it is often convenient to be able to change the term $12x^2$ easily. Thus, instead of inputting $f(x) = 12x^2$ directly and evaluating the weighted integral $12\int_{x_i}^{x_{i+1}} x^2 N_j dx$, one can be more general and interpolate $f(x)$ linearly as in $f(x) = f_i N_1 + f_{i+1} N_2$. This doesn't result in much loss of accuracy, since p is only accurate to this order in any case. Thus, assuming that $f(x)$ is known at the nodes (the f_i's),

$$\int_{x_i}^{x_{i+1}} \frac{\partial^2 p}{\partial x^2} w dx = 12 \int_{x_i}^{x_{i+1}} x^2 w dx$$

$$\rightarrow \quad p_i \int_{x_i}^{x_{i+1}} \frac{\partial N_1}{\partial x}\frac{\partial N_j}{\partial x} dx + p_{i+1} \int_{x_i}^{x_{i+1}} \frac{\partial N_2}{\partial x}\frac{\partial N_j}{\partial x} dx + f_i \int_{x_i}^{x_{i+1}} N_1 N_j dx + f_{i+1} \int_{x_i}^{x_{i+1}} N_2 N_j dx = \left[\frac{\partial p}{\partial x} N_j\right]_{x_i}^{x_{i+1}}$$

$$\rightarrow \quad p_i \frac{1}{L}\begin{bmatrix} +1 \\ -1 \end{bmatrix} + p_{i+1} \frac{1}{L}\begin{bmatrix} -1 \\ +1 \end{bmatrix} = \begin{bmatrix} -p'(x_i) \\ +p'(x_{i+1}) \end{bmatrix} - \frac{L}{6}\begin{bmatrix} 2f_i + f_{i+1} \\ f_i + 2f_{i+1} \end{bmatrix}$$

(3.28)

In the formula

$$q_h(x) = \frac{\partial p_h}{\partial x} + (x - x_c) f(x_c), \tag{3.29}$$

$f(x_c)$ can be evaluated from the linear interpolation of $f(x)$, i.e. simply the average of the nodal values, $(f_i + f_{i+1})/2$. One could also use the more specific expression $f(x_c) = 12x_c^2$.

Results are shown below for 3 elements. Note that the FE solution for the derivative q is discontinuous (very slightly so here) across the element boundaries.

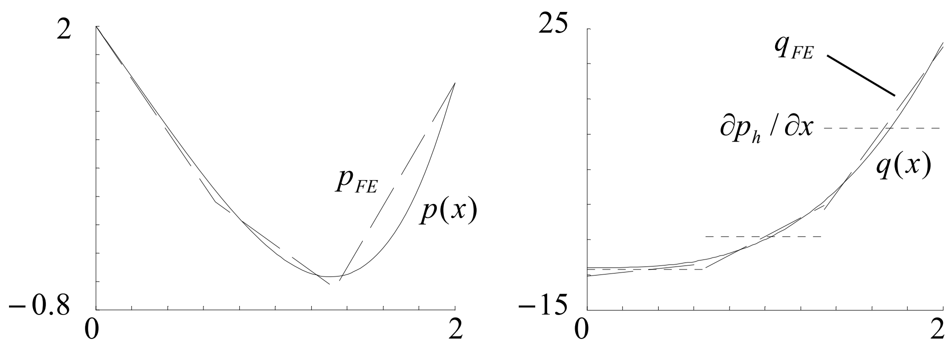

Figure 3.6: Non-Mixed Method solution to Eqn. 3.27

3.5 Problems

1. For the Penalty Method, show how the boundary term $[(\partial\omega/\partial x)p]$ leads to the final symmetric system (3.10). What is the accuracy obtained at the boundary, that is, what are the values of p_1, p_n in this case, in terms of $C = L\eta$, L, $\overline{p}$ and f_i?
2. For the standard mixed method, use (3.18) to write out the system of equations for the two-element mesh for the problem (3.16), the solution of which is $p_1 = p_2 = 1/3$.
3. Derive the system of equations (3.20).